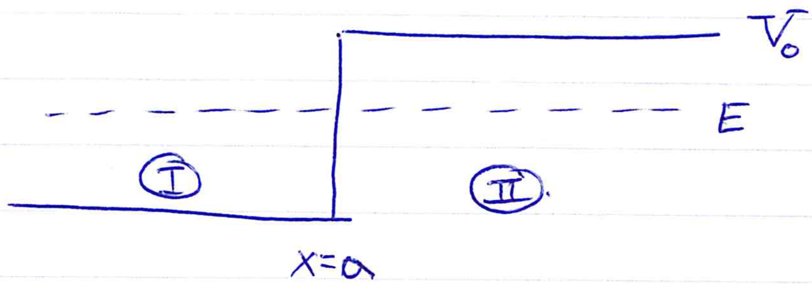
Assignment 4

PHYS 2380

Solutions.

# Assignment 4 PHYS 2380 KEY

1)




Probability current in Region I.

$$\Psi(x,t) = \left(Ae^{ik_1x} + Be^{-ik_1x}\right)e^{-iEt/\hbar} \qquad E = \frac{\hbar^2 k_1^2}{2m}$$

$$A = \frac{D}{2}\left(1 + i\frac{k_2}{k_1}\right) \qquad B = \frac{D}{2}\left(1 - i\frac{k_2}{k_1}\right)$$

$$J(x,t) = -\frac{i\hbar}{2m}\left[\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right].$$

The time dependent parts cancel, so $J = J(x)$.

$$\frac{\partial\Psi}{\partial x} = ik_1 A e^{ik_1x} - ik_1 B e^{-ik_1x}$$

$$\frac{\partial\Psi^*}{\partial x} = -ik_1 A^* e^{-ik_1x} + ik_1 B^* e^{ik_1x}$$

$$\Psi^* \frac{\partial \Psi}{\partial x} = \left(A^* e^{-ik_1x} + B^* e^{ik_1x}\right)\left(ik_1 A e^{ik_1x} - ik_1 B e^{-ik_1x}\right)$$

$$= ik_1 A^*A - ik_1 A^*B e^{-i2k_1x} + ik_1 AB^* e^{i2k_1x} - ik_1 B^*B$$

$$= ik_1\left[A^*A - B^*B - A^*Be^{-2ik_1x} + B^*Ae^{2ik_1x}\right]$$

Similarly

$$\frac{\partial \Psi^*}{\partial x}\Psi = ik_1\left[-A^*A + B^*B - A^*Be^{-2ik_1x} + B^*Ae^{i2k_1x}\right]$$

Then $J(x) = -\frac{i\hbar}{2m}\left[\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right]$

$$= \frac{\hbar k_1}{2m}\left[2A^*A - 2B^*B\right]$$

$$= \frac{\hbar k_1}{m}\left[A^*A - B^*B\right] \qquad \frac{p_1}{m} = \frac{\hbar k_1}{m} = v_1$$

$$= v_1\left[A^*A - B^*B\right]$$

$$J(x) = \underbrace{v_1 A^*A}_{\text{fwd current}} - \underbrace{v_1 B^*B}_{\text{reverse current}}$$

b.) $A^*A = \dfrac{D^*D}{4}\left(1 - i\dfrac{k_2}{k_1}\right)\left(1 + i\dfrac{k_2}{k_1}\right)$

$$B^*B = \frac{D^*D}{4}\left(1 + i\frac{k_2}{k_1}\right)\left(1 - i\frac{k_2}{k_1}\right)$$

$$J(x) = \emptyset$$

There is no flow of probability from region I - that is all probability is reflected from the boundary.

Recall our previous calculation

$$R = \frac{v_1}{v_1}\frac{A^*A}{B^*B} = 1!$$

(2) $\psi_0 = A_0 e^{-u^2/2}$

$\psi_1 = A_1 u e^{-u^2/2}$

$\psi_2 = A_2 (1-2u^2) e^{-u^2/2}$

a.) Orthogonality: $\int_{-\infty}^{\infty} \psi_g(u)\psi_s(u)\,du = \emptyset.$ $g \neq s.$

$g=0, s=1$ $A_1 A_0 \int_{-\infty}^{\infty} e^{-u^2/2} u e^{-u^2/2} du = A_1 A_0 \underbrace{\int_{-\infty}^{\infty} u e^{-u^2} du}_{\text{odd function over symmetric interval.}} = \emptyset.$

$g=0, s=2$ $A_2 A_0 \int_{-\infty}^{\infty} e^{-u^2/2}(1-2u^2)e^{-u^2/2} du$

$= A_2 A_0 \left[ \underbrace{\int_{-\infty}^{\infty} e^{-u^2} du}_{\sqrt{\pi}} - 2 \underbrace{\int_{-\infty}^{\infty} u^2 e^{-u^2} du}_{\frac{\sqrt{\pi}}{2}} \right]$ given in class

$= A_2 A_0 \left[ \sqrt{\pi} - 2 \cdot \frac{\sqrt{\pi}}{2} \right] = \emptyset.$

$g=1, s=2$ $A_1 A_2 \int_{-\infty}^{\infty} u e^{-u^2/2}(1-2u^2)e^{-u^2/2} du = A_1 A_2 \left[ \underbrace{\int_{-\infty}^{\infty} u e^{-u^2} du}_{\text{odd}} - 2 \underbrace{\int_{-\infty}^{\infty} u^3 e^{-u^2} du}_{\text{odd}} \right]$

$= \emptyset.$

b) $u = \sqrt{\alpha}\, x$, $\quad \dfrac{du}{dx} = \sqrt{\alpha} \quad$ or $\quad dx = \dfrac{du}{\sqrt{\alpha}}$

$$\frac{1}{\sqrt{\alpha}} \int_{-\infty}^{\infty} A_0^* e^{-u^2/2} A_0 e^{-u^2/2}\, du = \frac{1}{\sqrt{\alpha}} A_0^* A_0 \int_{-\infty}^{\infty} e^{-u^2} du = 1.$$

$$\frac{A_0^* A_0}{\sqrt{\alpha}} \left[ \sqrt{\pi} \right] = 1 \quad \text{or} \quad A_0^* A_0 = \sqrt{\frac{\alpha}{\pi}}.$$

$$\boxed{A_0 = \left(\frac{\alpha}{\pi}\right)^{1/4}.}$$

$$\frac{1}{\sqrt{\alpha}} \int_{-\infty}^{\infty} A_1^* u e^{-u^2/2} \cdot A_1 u e^{-u^2/2}\, du$$

$$= \frac{1}{\sqrt{\alpha}} A_1^* A_1 \int_{-\infty}^{\infty} u^2 e^{-u^2} du = \frac{1}{\sqrt{\alpha}} A_1^* A_1 \left[ \frac{\sqrt{\pi}}{2} \right] = 1.$$

$$A_1^* A_1 = 2\sqrt{\frac{\alpha}{\pi}}$$

$$\boxed{A_1 = 2^{1/2} \left(\frac{\alpha}{\pi}\right)^{1/4}}$$

$$\frac{1}{\sqrt{\alpha}} \int_{-\infty}^{\infty} A_2^* (1-2u^2) e^{-u^2/2} \cdot A_2 (1-2u^2) e^{-u^2/2}\, du$$

$$= \frac{1}{\sqrt{\alpha}} \int_{-\infty}^{\infty} A_2^* A_2 e^{-u^2} (1-2u^2)^2 du = \frac{A_2^* A_2}{\sqrt{\alpha}} \left[ \int_{-\infty}^{\infty} e^{-u^2} (1 - 4u^2 + 4u^4) du \right].$$

$$= \frac{A_2^* A_2}{\sqrt{\alpha}} \left[ \int_{-\infty}^{\infty} e^{-u^2} du - 4 \int_{-\infty}^{\infty} u^2 e^{-u^2} du + 4 \int_{-\infty}^{\infty} u^4 e^{-u^2} du \right].$$

$$= \frac{A_2^* A_2}{\sqrt{\alpha}} \left[ \sqrt{\pi} - 4 \frac{\sqrt{\pi}}{2} + \cancel{4} \cdot \frac{3\sqrt{\pi}}{\cancel{4}} \right]$$

$$= \frac{A_2^* A_2}{\sqrt{\alpha}} \left[ \sqrt{\pi} - 2\sqrt{\pi} + 3\sqrt{\pi} \right] = 2 A_2^* A_2 \sqrt{\frac{\pi}{\alpha}} = 1$$

$$A_2^2 = \frac{1}{2}\sqrt{\frac{\alpha}{\pi}}$$

$$\boxed{A_2 = \left(\frac{1}{2}\right)^{1/2} \left(\frac{\alpha}{\pi}\right)^{1/4}}$$

It's also ok if this is done in $u$-space, ie if the factor of $\alpha$ is absent ie $\alpha = 1$. Then the integrals will just have been set up over $du$ instead of $dx$. It is just a choice of units at the end of the day!

c.) Suppose $\Psi = \sqrt{\frac{3}{8}}\psi_0 + \sqrt{\frac{3}{8}}\psi_1 + \frac{1}{2}\psi_2$

Normalized wave function has

$$\int_{-\infty}^{\infty} |\Psi|^2 dx = 1$$ . Is this true?

$$\int_{-\infty}^{\infty} \Psi^* \Psi \, dx = N$$

$$\int_{-\infty}^{\infty} \left(\sqrt{\frac{3}{8}}\psi_0^* + \sqrt{\frac{3}{8}}\psi_1^* + \frac{1}{2}\psi_2^*\right)\left(\sqrt{\frac{3}{8}}\psi_0 + \sqrt{\frac{3}{8}}\psi_1 + \frac{1}{2}\psi_2\right) dx$$

$$= \int_{-\infty}^{\infty} \Big[ \frac{3}{8}\psi_0^*\psi_0 + \cancel{\frac{3}{8}\psi_0^*\psi_1} + \cancel{\frac{1}{2}\sqrt{\frac{3}{8}}\psi_0^*\psi_2}$$

$$+ \cancel{\frac{3}{8}\psi_1^*\psi_0} + \frac{3}{8}\psi_1^*\psi_1 + \cancel{\frac{1}{2}\sqrt{\frac{3}{8}}\psi_1^*\psi_2}$$

$$\cancel{\frac{1}{2}\sqrt{\frac{3}{8}}\psi_2^*\psi_0} + \cancel{\frac{1}{2}\sqrt{\frac{3}{8}}\psi_2^*\psi_1} + \frac{1}{4}\psi_2^*\psi_2 \Big] dx$$

All mixed terms vanish by orthogonality. Left with

$$\frac{3}{8}\int_{-\infty}^{\infty}\psi_0^*\psi_0 \, dx + \frac{3}{8}\int_{-\infty}^{\infty}\psi_1^*\psi_1 \, dx + \frac{1}{4}\int_{-\infty}^{\infty}\psi_2^*\psi_2 \, d$$

$$\frac{3}{8} + \frac{3}{8} + \frac{1}{4} = \frac{3}{8} + \frac{3}{8} + \frac{2}{8} = \frac{8}{8} = 1$$

$N = 1$ ✓

Yes, the preceeding wavefcn is normalized.

d.) Now, say $\Psi = c_0\psi_0 + c_1\psi_1 + c_2\psi_2$.

By the previous calculation, the orthogonal cross-terms vanish and we are left with

$$c_0^* c_0 \underbrace{\int_{-\infty}^{\infty} \psi_0^* \psi_0 \, dx}_{1} + c_1^* c_1 \underbrace{\int_{-\infty}^{\infty} \psi_1^* \psi_1 \, dx}_{1} + c_2^* c_2 \underbrace{\int \psi_2^* \psi_2}_{1}.$$

if this is normalized we have

$$c_0^* c_0 + c_1^* c_1 + c_2^* c_2 = 1$$

Yes, the weighted coefficients of c also have this property.

e.) The wavefunction is composed of 3 states each of which is weighted by $c_0, c_1, c_2$. The squared modulus of each give the probabilities of detecting the system in a given state, ~~[illegible]~~ such that

Prob of detecting system in $\psi_0$: $c_0^* c_0$
" " " " " $\psi_1$: $c_1^* c_1$
" " " " " $\psi_2$: $c_2^* c_2$.

The total probability must be 1 (ie, a system in superposition must be in one of these

3 a,) Classical turning points:

$n=1$

$$E_n = \left(n+\frac{1}{2}\right)\hbar\omega = \frac{3}{2}\hbar\omega.$$

$$V(x) = \frac{Cx^2}{2}$$ when these are equal,

$$\frac{Cx_T^2}{2} = \frac{3}{2}\hbar\omega.$$

$$x_T^2 = \frac{3\hbar\omega}{C}$$ from notes

$$\omega^2 = \frac{C}{m}$$ classical oscillato

$$x_T^2 = 3\frac{\hbar}{(Cm)^{1/2}} \qquad \omega = \sqrt{\frac{C}{m}}$$

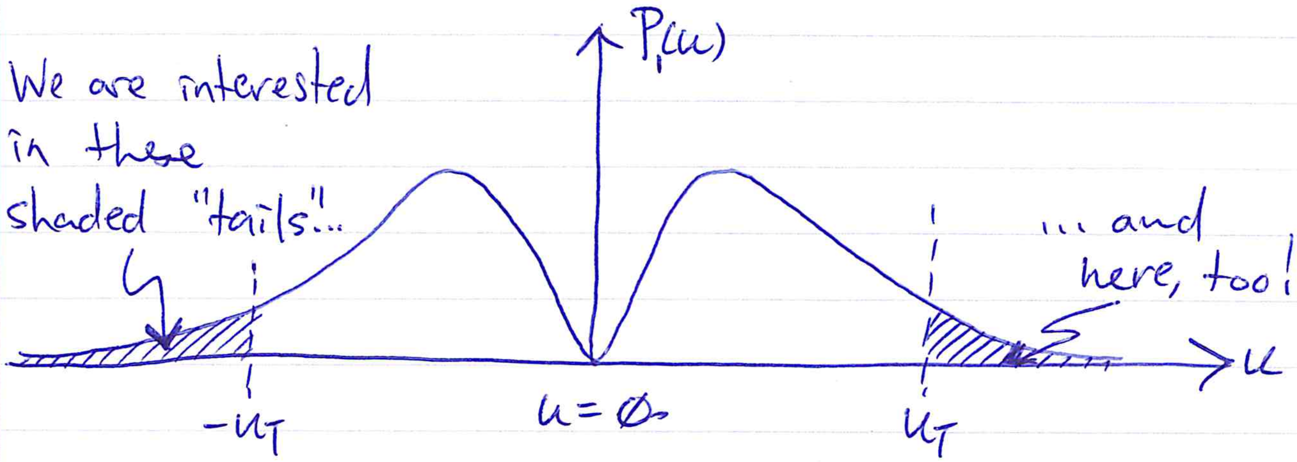
$$x_T = \pm\sqrt{3}\left(\frac{\hbar^{1/2}}{(cm)^{1/4}}\right)$$

$$u_T = \sqrt{\alpha}\, x_T = \pm\sqrt{3}\left(\frac{\hbar^{1/2}}{(cm)^{1/4}}\cdot\sqrt{\alpha}\right)$$

$$\alpha = \frac{\omega m}{\hbar} = \frac{(cm)^{1/2}}{\hbar} \qquad u_T = \pm\sqrt{3}\left(\frac{\hbar^{1/2}}{(cm)^{1/4}}\,\frac{(cm)^{1/4}}{\hbar^{1/2}}\right)$$

$$\boxed{u_T = \pm\sqrt{3}}$$

b.) Probability to find particle outside classical turning points:




$$du = \sqrt{\alpha}\, dx \quad , \qquad dx = \frac{du}{\sqrt{\alpha}}.$$

$$P = \int_{-\infty}^{-u_T} \psi_1^* \psi_1 \, dx + \int_{u_T}^{\infty} \psi_1^* \psi_1 \, dx$$

$$= \frac{1}{\sqrt{\alpha}} \int_{-\infty}^{-u_T} A_1^2 u^2 e^{-u^2} du + \frac{1}{\sqrt{\alpha}} \int_{u_T}^{\infty} A_1^2 u^2 e^{-u^2} du$$

Integrands are even, so combine these.

$$P = \frac{2}{\sqrt{\alpha}} \int_{u_T}^{\infty} A_1^2 u^2 e^{-u^2} du = \frac{2A_1^2}{\sqrt{\alpha}} \int_{\sqrt{3}}^{\infty} u^2 e^{-u^2} du$$

From the table of integral values on the website,

$$I = \int_{\sqrt{3}}^{\infty} u^2 e^{-u^2} du = 0.0495$$

Then using $A_1^2 = 2\sqrt{\frac{\alpha}{\pi}}$, we find

$$P = \frac{2A_1^2 I}{\sqrt{\alpha}} = \frac{2}{\sqrt{\alpha}} \cdot 2\sqrt{\frac{\alpha}{\pi}} I = \frac{4}{\sqrt{\pi}} I$$

$$P = \frac{4}{\sqrt{\pi}} \cdot 0.0495$$

$$\boxed{P = 0.112}$$

# OPTIONAL

Ⓡ

If interested, you can integrate by parts to make some progress

$$\int_{\sqrt{3}}^{\infty} u^2 e^{-u^2} du = \int_{\sqrt{3}}^{\infty} \underset{\uparrow\, f}{u} \underbrace{\left(u e^{-u^2}\right)}_{dg} du$$

set $f = u \implies df = du.$

$dg = u e^{-u^2} du \implies g = -\frac{1}{2} e^{-u^2}$

$$= fg - \int g\, df$$

$$= -\frac{u e^{-u^2}}{2}\Big|_{\sqrt{3}}^{\infty} - \int_{\sqrt{3}}^{\infty} \left(-\frac{1}{2} e^{-u^2}\right) du$$

$$= -\frac{u e^{-u^2}}{2}\Big|_{\sqrt{3}}^{\infty} + \frac{1}{2} \underbrace{\int_{\sqrt{3}}^{\infty} e^{-u^2} du}_{\text{this is defined as a complementary error function (yuck)}}$$

$$= -\frac{u e^{-u^2}}{2}\Big|_{\sqrt{3}}^{\infty} + \frac{1}{2} \cdot \frac{\sqrt{\pi}}{2} \operatorname{erfc}(\sqrt{3})$$

where $\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-x^2} dx.$

Left with $\boxed{= \frac{\sqrt{3}}{2} e^{-3} + \frac{\sqrt{\pi}}{4} \operatorname{erfc}(\sqrt{3})}$

4 $\Psi_0 = A_0 e^{-u^2/2} e^{-iE_0 t/\hbar}$.

a.) $\langle x \rangle = \int_{-\infty}^{\infty} \Psi_0^* x \Psi_0 \, dx \qquad u = \sqrt{\alpha}\, x, \quad \frac{u}{\sqrt{\alpha}} = x$

$\frac{du}{\sqrt{\alpha}} = dx.$

$= \frac{1}{\alpha} \int_{-\infty}^{\infty} A_0^2 e^{-u^2/2} \cancel{e^{iE_0t/\hbar}} \cdot u e^{-u^2/2} \cancel{e^{-iE_0t/\hbar}} du$

$= \frac{A_0^2}{\alpha} \underbrace{\int_{-\infty}^{\infty} u e^{-u^2} du}$

odd function over symmetric interval

$\langle x \rangle = \emptyset$

$\langle x^2 \rangle = \int_{-\infty}^{\infty} \Psi_0^* x^2 \Psi_0 \, dx \qquad x^2 = \frac{u^2}{\alpha}, \quad dx = \frac{du}{\sqrt{\alpha}}$

$= \frac{1}{\alpha^{3/2}} \int_{-\infty}^{\infty} A_0 e^{-u^2/2} \cancel{e^{iE_0t/\hbar}} u^2 A_0 e^{-u^2/2} \cancel{e^{-iE_0t/\hbar}} du$

$= \frac{1}{\alpha^{3/2}} A_0^2 \underbrace{\int_{-\infty}^{\infty} u^2 e^{-u^2} du}$

$\frac{1}{\alpha^{3/2}} A_0^2 \frac{\sqrt{\pi}}{2} = \frac{1}{\alpha^{3/2}} \sqrt{\frac{\alpha}{\pi}} \cdot \frac{\sqrt{\pi}}{2} = \frac{1}{2\alpha}.$

$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}, \quad \Delta x = \sqrt{\langle x^2 \rangle} = \frac{1}{\sqrt{2\alpha}}.$

b.) $p_{op} = -i\hbar \frac{\partial}{\partial x}$ , $\frac{\partial}{\partial x} \rightarrow \sqrt{\alpha}\frac{\partial}{\partial u}$.

$$\langle p\rangle = \int_{-\infty}^{\infty} \Psi_0^* p_{op} \Psi_0 \, dx = -i\hbar \int_{-\infty}^{\infty} A_0 e^{-u^2/2} e^{+iE_0t/\hbar} \cdot \frac{\partial}{\partial u}\left(A_0 e^{-u^2/2} e^{-i\ldots}\right.$$

$$\frac{\partial \Psi_0}{\partial u} = \left(-u e^{-u^2/2}\right) e^{-iE_0 t/\hbar}$$

so $\langle p\rangle = i\hbar A_0^2 \int_{-\infty}^{\infty} u e^{-u^2} du = \varnothing$ by symmet (odd fcn)

$$\langle p^2\rangle = \int_{-\infty}^{\infty} \Psi_0^* p_{op}^2 \Psi_0 \, dx = -\frac{\hbar^2 \alpha}{\sqrt{\alpha}} A_0^2 \int_{-\infty}^{\infty} e^{-u^2/2} \frac{\partial^2}{\partial u^2}\left(e^{-u^2/2}\right) d$$

$$\frac{\partial^2 \Psi_0}{\partial u^2} = u^2 e^{-u^2/2} - e^{-u^2/2}$$

$$\langle p^2\rangle = -\frac{\hbar^2 \alpha}{\sqrt{\alpha}} \left(\frac{\alpha}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} e^{-u^2/2}\left(u^2 e^{-u^2/2} - e^{-u^2/2}\right) du$$

$$\langle p^2\rangle = \frac{\hbar^2 \alpha}{\sqrt{\pi}} \left[\int_{-\infty}^{\infty} e^{-u^2} du - \int_{-\infty}^{\infty} u^2 e^{-u^2} du\right]$$

$$= \frac{\hbar^2 \alpha}{\sqrt{\pi}} \left[\sqrt{\pi} - \frac{\sqrt{\pi}}{2}\right] = \frac{\hbar^2 \alpha}{2}$$

$$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\langle p^2\rangle}$$

$$\Delta p = \hbar \sqrt{\frac{\alpha}{2}}$$

c.) $\Delta x \Delta p = \frac{1}{\sqrt{2\alpha}} \cdot \hbar\sqrt{\frac{\alpha}{2}} = \frac{\hbar}{2}$

$\Delta x \Delta p = \frac{\hbar}{2}$ minimum uncertainty! Lower bound (equality) in H.U.P satisfied here.

d.) $\Delta p = \hbar\sqrt{\frac{\alpha}{2}}$, $\Delta E = \frac{\Delta p^2}{2m} = \frac{\hbar^2 \alpha}{4m}$

$$\alpha = \frac{\omega m}{\hbar}$$

so $\Delta E = \frac{\hbar^2 \alpha}{4m} = \frac{\hbar^2}{4m} \cdot \frac{\omega m}{\hbar} = \frac{\hbar\omega}{4} = \frac{h\nu}{4}$

ground state $E_0 = \left(\frac{1}{2}\right) h\nu$ so $\Delta E = \frac{E_0}{2}$

This is consistent with the result we found in class on Monday March 20 using the kinetic energy operator. In that example case we proved the equipartition theorem from classical mechanics holds on average for the quantum case. This makes intuitive sense since quantum mechanics must "average out" to give back classical physics on large scales.