PHYS 2380

Assignment 3

Answer key

1

$\lambda = 0.1 \text{ nM}$

$$\lambda = \frac{h}{p} = \frac{h}{(2mE)^{1/2}}$$

so $E = \dfrac{p^2}{2m} = \dfrac{h^2}{2m\lambda^2}$

$$E = \frac{6.63 \times 10^{-34} \text{ J}\cdot\text{s}}{2(9.11 \times 10^{-31} \text{ kg})(0.1 \times 10^{-9} \text{ nM})^2}$$

$$\boxed{E = 2.41 \times 10^{-17} \text{ J} = 151 \text{ eV}}$$

② $\lambda = 694.3\ nM$

$n=2$

$\downarrow$

$n=1$

$$\Delta E = E_{n_2} - E_{n_1}$$

$$= \frac{4\pi^2\hbar^2}{2m_eL^2} - \frac{\pi^2\hbar^2}{2m_eL^2}$$

$$\Delta E = \frac{hc}{\lambda}$$

$$= \frac{3\pi^2\hbar^2}{2m_eL^2}$$

$$\frac{hc}{\lambda} = \frac{3\pi^2\hbar^2}{2m_eL^2}$$

$$L^2 = \frac{3\pi^2\hbar^2}{2m_e}\frac{\lambda}{hc}$$

$$= \frac{3\pi^2 h^2 \lambda}{4\pi^2(2m_e hc)}$$

$$= \frac{3}{8}\frac{h\lambda}{m_e c}$$

$$L = \left(\frac{3}{8}\frac{hc\lambda}{m_e c^2}\right)^{1/2}$$

$E_e = 0.511\ MeV$
$= 5.11 \times 10^5\ eV$

$$= \left(\frac{3}{8}\frac{(1240\ eV\cdot nM)(694.3)}{5.11\times10^5\ eV}\right)^{1/2}$$

$$\boxed{L = 0.795\ nM}$$

③ $n=5$

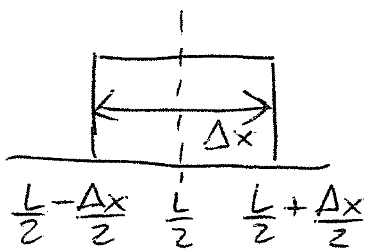
a.) $\psi(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{5\pi x}{L}\right)$

$$P = \int_{0.2L}^{0.4L} \frac{2}{L} \sin^2\left(\frac{5\pi x}{L}\right) dx = \frac{2}{L} \int_{0.2L}^{0.4L} \sin^2\left(\frac{5\pi x}{L}\right) dx$$

$$= \frac{1}{5}\left(\frac{-\cos\left(\frac{5\pi x}{L}\right)\sin\left(\frac{5\pi x}{L}\right)}{\pi} + \frac{5x}{L}\right)\Bigg|_{0.2L}^{0.4L}$$

$$\boxed{P = 0.2}$$

b.)



$$P = \int_{\frac{L-\Delta x}{2}}^{\frac{L+\Delta x}{2}} \frac{2}{L} \sin^2\left(\frac{5\pi x}{L}\right) dx$$

$$= \int_{\frac{0.99L}{2}}^{\frac{1.01L}{2}} \frac{2}{L} \sin^2\left(\frac{5\pi x}{L}\right) dx$$

$\Delta x = 0.01L$

$\frac{L}{2} - \frac{\Delta x}{2} = \frac{0.99L}{2}$

$$= \frac{1}{5}\left(\frac{-\cos\left(\frac{5\pi x}{L}\right)\sin\left(\frac{5\pi x}{L}\right)}{\pi} + \frac{5x}{L}\right)\Bigg|_{\frac{0.99L}{2}}^{\frac{1.01L}{2}}$$

$\frac{L}{2} + \frac{\Delta x}{2} = \frac{1.01L}{2}$

$$\boxed{P = 0.02}.$$

④ Find the Fourier transform of $g(k) = e^{-|k|/a}$. Deal with the absolute value by breaking the domain of integration in half using:

$$\Psi = \int_{-\infty}^{\infty} A e^{-|k|/a} e^{+ikx} dk = \underbrace{\int_{-\infty}^{0} A e^{k/a} e^{ikx} dk}_{Ⓐ} + \underbrace{\int_{0}^{\infty} A e^{-k/a} e^{ikx} dk}_{Ⓑ}.$$

$$Ⓐ: A\int_{-\infty}^{0} e^{k/a} e^{ikx} dx = A\int_{-\infty}^{0} e^{k(\frac{1}{a}+ix)} dk = \frac{Ae^{k\beta}}{\beta}\Big|_{-\infty}^{0}$$

$$= \frac{A}{\beta} = \frac{A}{(\frac{1}{a}+ix)}.$$

$$Ⓑ: \int_{0}^{\infty} e^{-k/a} e^{ikx} dk = \int_{0}^{\infty} e^{-k(\frac{1}{a}-ix)} dk = -\frac{e^{-k\beta}}{\beta}\Big|_{0}^{\infty} = \frac{1}{\beta}$$

$$= \frac{A}{(\frac{1}{a}-ix)}$$

$$\psi = A\left(\frac{1}{(\frac{1}{a}+ix)} + \frac{1}{(\frac{1}{a}-ix)}\right) = A\left(\frac{(\frac{1}{a}-ix)}{(\frac{1}{a}+ix)(\frac{1}{a}-ix)} + \frac{(\frac{1}{a}+ix)}{(\frac{1}{a}+ix)(\frac{1}{a}-ix)}\right)$$

$$\psi = \frac{2A}{a(\frac{1}{a^2}+x^2)}$$

b.) Normalize the wave function to find the constant A.

$$\int_{-\infty}^{\infty} \psi^* \psi \, dx = 1$$

$$= \frac{4A^2}{a^2}\int_{-\infty}^{\infty} \frac{1}{(\frac{1}{a^2}+x^2)^2} dx = 1 \qquad x = \frac{1}{a}\tan\theta, \quad dx = \frac{1}{a}\sec^2\theta \, d\theta$$

$$= \frac{4A^2}{a^2}\int_{-\pi/2}^{\pi/2} \frac{1}{a}\frac{\sec^2\theta \, d\theta}{(\frac{1}{a^2}+\frac{1}{a^2}\tan^2\theta)^2} = \frac{4A^2}{a^3}\cdot a^4 \int_{-\pi/2}^{\pi/2} \frac{\sec^2\theta \, d\theta}{(1+\tan^2\theta)^2}$$

⑤ a) Gaussian centered at k_0:

$$g(k) = A \exp\left(-\frac{(k-k_0)^2}{4\sigma_k^2}\right) \qquad \text{Set } k' = k - k_0, \quad dk' = dk$$

$$\psi = \int_{-\infty}^{\infty} A e^{-\frac{(k-k_0)^2}{4\sigma_k^2}} e^{ikx}\, dk = \int_{-\infty}^{\infty} A e^{-\frac{k'^2}{4\sigma_k^2}} e^{i(k'+k_0)x}\, dk'$$

$$= A \int_{-\infty}^{\infty} e^{-\alpha\left(k'^2 - \frac{ik'x}{\alpha}\right)} e^{ik_0x}\, dk' \qquad \alpha = \frac{1}{4\sigma_k^2}$$

$$= A e^{ik_0x} \int_{-\infty}^{\infty} e^{-\alpha\left(k'^2 - \frac{ik'x}{\alpha}\right)}\, dk'$$

Complete the square in the exponent:

$$\left(k' - \frac{ix}{2\alpha}\right)^2 - \left(\frac{ix}{2\alpha}\right)^2$$

set $y = k' - \frac{ix}{2\alpha}$, $dy = dk'$

$$= A e^{ik_0x} \int_{-\infty}^{\infty} e^{-\alpha\left(k' - \frac{ix}{2\alpha}\right)^2} e^{+\alpha\left(\frac{ix}{2\alpha}\right)^2}\, dk'$$

$$= A e^{ik_0x} e^{-\frac{x^2}{4\alpha}} \underbrace{\int_{-\infty}^{\infty} e^{-\alpha y^2}\, dy}_{\sqrt{\frac{\pi}{\alpha}}}$$

$$= A e^{ik_0x} e^{-\frac{x^2}{4\alpha}} \sqrt{\frac{\pi}{\alpha}} = A e^{ik_0x} \sqrt{\pi} \cdot 2\sigma_k\, e^{-x^2\sigma_k^2}$$

Set $\boxed{\sigma_x^2 = \frac{1}{4\sigma_k^2}}$, $\psi = B e^{ik_0x} e^{-\frac{x^2}{4\sigma_x^2}}$

b) Normalize:

$$\int_{-\infty}^{\infty} \psi^* \psi \, dx = \int_{-\infty}^{\infty} B^* B e^{-ik_0 x} e^{-\frac{x^2}{4\sigma_x^2}} e^{ik_0 x} e^{-\frac{x^2}{4\sigma_x^2}} dx$$

$$= \int_{-\infty}^{\infty} B^* B e^{-\frac{x^2}{2\sigma_x^2}} dx \quad = B^* B \underbrace{\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma_x^2}} dx}_{\sqrt{\frac{\pi}{\alpha}} = \sqrt{2\pi \sigma_x^2}}$$

$$B^* B \sqrt{2\pi \sigma_x^2} = 1$$

$$B^2 = \frac{1}{(2\pi \sigma_x^2)^{1/2}}$$

$$\boxed{B = \frac{1}{(2\pi \sigma_x^2)^{1/4}}}$$

c.) For the Gaussian centered at $\emptyset$:

$$\psi = A e^{-\frac{x^2}{4\sigma_x^2}} \qquad \frac{\partial \psi}{\partial x} = A \left(\frac{-2x}{4\sigma_x^2}\right) e^{-\frac{x^2}{4\sigma_x^2}}$$

$$\langle p \rangle = \int_{-\infty}^{\infty} \psi^* p_{op} \psi \, dx = (-i\hbar) \int_{-\infty}^{\infty} A^2 \left(\frac{-2x}{4\sigma_x^2}\right) e^{-\frac{x^2}{2\sigma_x^2}} dx$$

$$= \frac{i\hbar 2}{4\sigma_x^2} A^2 \int_{-\infty}^{\infty} \underbrace{x e^{-\frac{x^2}{2\sigma_x^2}}}_{\text{odd function!}} dx$$

$$\langle p \rangle = 0$$

$$4A^2 a \int_{-\pi/2}^{\pi/2} \frac{\sec^2\theta}{\sec^4\theta}\, d\theta = 4A^2 a \int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta$$

$$= 4A^2 a \left[\frac{1}{2}\cos\theta\sin\theta + \frac{\theta}{2} \right] \Bigg|_{-\pi/2}^{\pi/2} = 2A^2 a \pi$$

$$2A^2 a \pi = 1$$

$$A^2 = \frac{1}{2a\pi} \qquad A = \sqrt{\frac{1}{2\pi} \cdot \frac{1}{a}}.$$

d.) For Gaussian centered at k_0:

$$\langle p \rangle = \int_{-\infty}^{\infty} \psi^* p_{op} \psi \, dx = \int_{-\infty}^{\infty} B^2 e^{-ik_0 x} e^{-\frac{x^2}{4\sigma_x^2}} \left(-i\hbar \frac{\partial}{\partial x} \left[e^{+ik_0 x} e^{-\frac{x^2}{4\sigma_x^2}} \right] \right) dx$$

$$= B^2(-i\hbar) \int_{-\infty}^{\infty} e^{-ik_0 x} e^{-\frac{x^2}{4\sigma_x^2}} \underbrace{\frac{\partial}{\partial x}\left(e^{ik_0 x} e^{-\frac{x^2}{4\sigma_x^2}} \right)}_{(ik_0) e^{ik_0 x} e^{-\frac{x^2}{4\sigma_x^2}} + \left(-\frac{2x}{4\sigma_x^2}\right) e^{ik_0 x} e^{-\frac{x^2}{4\sigma_x^2}}} dx$$

$$= B^2(-i\hbar) \int_{-\infty}^{\infty} \left[(ik_0) \underbrace{e^{-\frac{x^2}{2\sigma_x^2}}}_{\sqrt{2\pi\sigma_x^2}} - \underbrace{\frac{1}{4\sigma_x^2} x e^{-\frac{x^2}{4\sigma_x^2}}}_{0} \right] dx$$

by symmetry, odd function

$$= B^2(-i\hbar)(ik_0)\sqrt{2\pi\sigma_x^2}$$

previously, $B^2 = \frac{1}{(2\pi\sigma_x^2)^{1/2}}$

$$\langle p \rangle = -i^2 \hbar k_0$$

$$\boxed{\langle p \rangle = \hbar k_0}$$

e.) The expectation value for momentum gives the peak → "symmetry axis" of the distribution in p.