2017 PHYS 2380
MID-TERM
KEY

① $$u(\lambda) = \frac{8\pi hc}{\lambda^5}\left(\frac{1}{e^{hc/\lambda k_b T}-1}\right)$$

$$\frac{du}{d\lambda} = 0 = \cancel{\frac{8\pi hc}{\lambda^5}}\left[-\left(e^{hc/\lambda k_b T}-1\right)^{-2}\left(-\frac{hc}{\lambda^2 k_b T}\right)e^{hc/\lambda k_b T}\right]$$

$$- \cancel{8\pi hc}\left(\frac{5}{\lambda^{\cancel{6}}}\right)\cancel{\frac{1}{\left(e^{hc/\lambda k_b T}-1\right)}}$$

$$= +\left(e^{hc/\lambda k_b T}-1\right)^{-1}\frac{hc\,e^{hc/\lambda k_b T}}{\lambda^2 k_b T} - \frac{5}{\cancel{\lambda}}$$

$$\frac{e^{hc/\lambda k_b T}}{e^{hc/\lambda k_b T}-1} = 1$$

so $$5 = \frac{hc}{\lambda k_b T}$$

or

$$\boxed{\lambda T = \frac{hc}{5k_b}}$$

2) a.) $\Psi = \Psi_1 + \Psi_2$

$= \psi_1 \exp(i\phi_1) + \psi_2 \exp(i\phi_2)$ Superposition principle

b.) $P(x) = \Psi^* \Psi = (|\psi_1|^* e^{-i\phi_1} + |\psi_2|^* e^{-i\phi_2})(|\psi_1| e^{i\phi_1} + |\psi_2| e^{i\phi_2})$

$= |\psi_1|^* |\psi_1| + |\psi_1|^* |\psi_2| e^{-i\phi_1} e^{i\phi_2} + |\psi_2|^* |\psi_1| e^{-i\phi_2} e^{i\phi_1} + |\psi_2|^* |\psi_2|$

$= |\psi_1|^2 + |\psi_2|^2 + |\psi_1|^* |\psi_2| e^{i(\phi_2 - \phi_1)} + |\psi_2|^* |\psi_1| e^{-i(\phi_2 - \phi_1)}$

$= |\psi_1|^2 + |\psi_2|^2 + \underbrace{2|\psi_1||\psi_2| \cos(\phi_2 - \phi_1)}_{\text{interference term.}}$

3.) Copenhagen Interpretation:

1) A wave function represents the state of a system. It contains all information about anything that can be known about a system before an observation. The wavefunction evolves smoothly in time.

2.) Certain properties cannot be known simultaneous for a given system (Heisenbergs uncertainty Principle), Momentum is meaningless for a perfectly localized particle.

3.) During an observation, the system must interact with a laboratory device. When that device makes a measurement, the wavefunction collapses and reduces irreversibly to an eigenfunction corresponding to a particular eigenvalue.

4.) The results provided by measuring devices are classical and must be stated in classical terms

5.) The Born Rule: Wavefunctions give a probabilistic description of an experimental outcome.

6.) Complementarity rule (Bohr): The wavefunction demonstrates a fundamental wave-particle duality that is a necessary feature of nature.

7.) Correspondence rule (Bohr + Heisenberg): When quantum numbers are large, the behaviour of a system must reproduce classical predictions.

4 a) $E_{op} = i\hbar \frac{\partial}{\partial t}$

$$E_{op}\psi_2 = i\hbar\frac{\partial}{\partial t}\left[A\sin\left(\frac{2\pi x}{a}\right)e^{-4iE_1t/\hbar}\right]$$

$$= i\hbar A\sin\left(\frac{2\pi x}{a}\right)e^{-4iE_1t/\hbar}\left(-\frac{4iE_1}{\hbar}\right)$$

$$= -4E_1 i^2 A\sin\left(\frac{2\pi x}{a}\right)e^{-4iE_1t/\hbar}$$

$$= 4E_1 A\sin\left(\frac{2\pi x}{a}\right)e^{-4iE_1t/\hbar}$$

$$= 4E_1\psi_2$$ Yes eigenfunction!

Eigenvalue $= 4E_1$

$$E_{op}\psi_3 = i\hbar\frac{\partial}{\partial t}\left[B\cos\left(\frac{3\pi x}{a}\right)e^{-9iEt/\hbar}\right]$$

$$= \cancel{i\hbar}\left(\frac{-9\cancel{i}E}{\cancel{\hbar}}\right)B\cos\left(\frac{3\pi x}{a}\right)e^{-9iEt/\hbar}$$

$$= 9E\psi_3$$ Yes eigenfunction!

eigenvalue $= 9E_1$

b.) $$E_{op}\psi = i\hbar\frac{\partial}{\partial t}\left[\frac{1}{\sqrt{a}}\left(\sin\left(\frac{2\pi x}{a}\right)e^{-4iE_1t/\hbar} + \cos\left(\frac{3\pi x}{a}\right)e^{-9iE_1t/\hbar}\right)\right]$$

$$= 4E_1\frac{1}{\sqrt{a}}\sin\left(\frac{2\pi x}{a}\right)e^{-4iEt/\hbar} + 9E_1\frac{1}{\sqrt{a}}\cos\left(\frac{3\pi x}{a}\right)e^{-9iEt/\hbar}$$

<u>Not</u> an eigenfunction.

a) $\Psi(x) = \frac{1}{\sqrt{a}}\left[\sin\left(\frac{2\pi x}{a}\right)e^{-4iEt/\hbar} + \cos\left(\frac{3\pi x}{a}\right)e^{-9iEt/\hbar}\right].$

$$P(x) = \Psi^*\Psi = \frac{1}{a}\left[\sin\left(\frac{2\pi x}{a}\right)e^{4iEt/\hbar} + \cos\left(\frac{3\pi x}{a}\right)e^{9iEt/\hbar}\right]$$

$$\times\left[\sin\left(\frac{2\pi x}{a}\right)e^{-4iEt/\hbar} + \cos\left(\frac{3\pi x}{a}\right)e^{-9iEt/\hbar}\right]$$

$$= \frac{1}{a}\left[\sin^2\left(\frac{2\pi x}{a}\right) + \sin\left(\frac{2\pi x}{a}\right)\cos\left(\frac{3\pi x}{a}\right)e^{-5iEt/\hbar}\right.$$

$$\left. + \sin\left(\frac{2\pi x}{a}\right)\cos\left(\frac{3\pi x}{a}\right)e^{5iEt/\hbar} + \cos^2\left(\frac{3\pi x}{a}\right)\right].$$

$$P(x) = \frac{1}{a}\left[\sin^2\left(\frac{2\pi x}{a}\right) + \cos^2\left(\frac{3\pi x}{a}\right) + 2\sin\left(\frac{2\pi x}{a}\right)\cos\left(\frac{3\pi x}{a}\right)\cos\left(\frac{5Et}{\hbar}\right.\right.$$

$$d_1)\ \langle x\rangle = \frac{1}{a}\int_{-a/2}^{a/2} x\left[\sin^2\left(\frac{2\pi x}{a}\right) + \cos^2\left(\frac{3\pi x}{a}\right) + 2\cos\left(\frac{5Et}{\hbar}\right)\sin\left(\frac{2\pi x}{a}\right)\cos\left(\frac{3\pi}{a}\right.\right.$$

$$= \frac{1}{a}\, 2\cos\left(\frac{5Et}{\hbar}\right)\int_{-a/2}^{a/2} x\sin\left(\frac{2\pi x}{a}\right)\cos\left(\frac{3\pi x}{a}\right)dx$$

$$= \frac{2}{a}\cos\left(\frac{5Et}{\hbar}\right)\left[\frac{-24}{25}\frac{a^2}{\pi^2}\right]$$

$$m\frac{d}{dt}\langle x\rangle = -\frac{2m}{a}\frac{5E}{\hbar}\sin\left(\frac{5Et}{\hbar}\right)\left[\frac{-24}{25}\frac{a^2}{\pi^2}\right] \qquad E = \frac{\hbar^2\pi^2}{2ma^2}$$

$$= \frac{\cancel{2ma^2}}{\cancel{\hbar}}\left(\frac{\cancel{\hbar^2}\cancel{\pi^2}}{\cancel{2ma^2}}\right)\sin\left(\frac{5Et}{\hbar}\right)\left[\frac{+24}{5}\frac{1}{a\cancel{\pi^2}}\right]$$

$$m\frac{d\langle x\rangle}{dt} = \frac{+24\hbar}{5a\cancel{\pi^2}}\sin\left(\frac{5Et}{\hbar}\right)$$

$$\langle p \rangle = \int_{-\infty}^{\infty} \psi^* p_{op} \psi \, dx \qquad p_{op} = -i\hbar \frac{\partial}{\partial x}$$

$$p_{op}\psi = \frac{-i\hbar}{\sqrt{a}} \left[\frac{2\pi}{a} \cos\left(\frac{2\pi x}{a}\right) e^{-4iEt/\hbar} - \frac{3\pi}{a} \sin\left(\frac{3\pi x}{a}\right) e^{-9iEt/\hbar} \right]$$

$$\psi^* p_{op} \psi = \frac{-i\hbar}{a} \left[\sin\left(\frac{2\pi x}{a}\right) e^{4iEt/\hbar} + \cos\left(\frac{3\pi x}{a}\right) e^{9iEt/\hbar} \right]$$

$$\times \left[\frac{2\pi}{a} \cos\left(\frac{2\pi x}{a}\right) e^{-4iEt/\hbar} - \frac{3\pi}{a} \sin\left(\frac{3\pi x}{a}\right) e^{-9iEt/\hbar} \right]$$

$$= \frac{-i\hbar}{a} \int_{-a/2}^{a/2} \left[\frac{2\pi}{a} \sin\left(\frac{2\pi x}{a}\right) \cos\left(\frac{2\pi x}{a}\right) \cancel{- \frac{3\pi}{a} \sin^2\left(\frac{3\pi x}{a}\right) e^{-5iEt/\hbar}} \right.$$

$$+ \frac{2\pi}{a} \cos\left(\frac{2\pi x}{a}\right) \cos\left(\frac{3\pi x}{a}\right) e^{5iEt/\hbar}$$

$$- \frac{3\pi}{a} \cos\left(\frac{3\pi x}{a}\right) \sin\left(\frac{3\pi x}{a}\right) - \frac{3\pi}{a} \sin\left(\frac{2\pi x}{a}\right) \sin\left(\frac{3\pi x}{a}\right) e^{-5iEt/\hbar}$$

$$= \frac{-i\hbar}{a} \left[\int_{-a/2}^{a/2} \frac{2\pi}{a} \cos\left(\frac{2\pi x}{a}\right) \cos\left(\frac{3\pi x}{a}\right) e^{5iEt/\hbar} dx \right.$$

$$\left. - \int_{-a/2}^{a/2} \frac{3\pi}{a} \sin\left(\frac{2\pi x}{a}\right) \sin\left(\frac{3\pi x}{a}\right) e^{-5iEt/\hbar} dx \right]$$

$$= \frac{-i\hbar}{a} \left[\frac{2\cancel{\pi}}{\cancel{a}} e^{5iEt/\hbar} \frac{6}{5} \frac{\cancel{a}}{\cancel{\pi}} - \frac{3\cancel{\pi}}{\cancel{a}} e^{-5iEt/\hbar} \frac{4}{5} \frac{\cancel{a}}{\cancel{\pi}} \right]$$

$$= \frac{-2i^2\hbar}{a} \frac{12}{5} \left(\frac{e^{5iEt/\hbar} - e^{-5iEt/\hbar}}{2i} \right) = -i^2 \frac{\hbar}{a} \frac{24}{5} \sin\left(\frac{5Et}{\hbar}\right)$$

$$\langle p \rangle = \frac{\hbar}{a} \frac{24}{5} \sin\left(\frac{5Et}{\hbar}\right) = m \frac{d\langle x \rangle}{dt}$$

5.) $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$

Suppose $E > V_0$, $V(x) = V_0$.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = (E - V_0)\psi$$

$$\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}(E-V_0)\psi$$

set $k^2 = \frac{2m}{\hbar^2}(E - V_0)$

$$\frac{d^2\psi}{dx^2} = -k^2\psi$$

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0$$

$$(d^2 + k^2)\psi = 0$$

$$d = \pm ik$$

$$\boxed{\psi = Ae^{ikx} + Be^{-ikx}}$$

b.) $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$

Suppose $E < V_0$, $V(x) = V_0$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = (E - V_0)\psi$$

$$\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}\left(E - V_0\right)\psi$$

$$\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}\left(V_0 - E\right)\psi$$

$$k^2 = \frac{2m}{\hbar^2}\left(V_0 - E\right)$$

$$\frac{d^2\psi}{dx^2} = k^2\psi$$

$$\frac{d^2\psi}{dx^2} - k^2\psi = 0$$

$$(d^2 - k^2)\psi = 0$$

$$d = \pm k$$

$$\boxed{\psi = Ae^{kx} + Be^{-kx}}$$