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1. $T = 4900\,K$ $\quad R = 6.96 \times 10^{9}\,m$

(a) Energy flux: $\sigma T^4 = (5.67 \times 10^{-8}\,W/m^2 \cdot K^4)(4900K)^4$

$= 3.269 \times 10^{7}\,W/m^2$

Total Power radiated: $(\sigma T^4)\,4\pi R^2$

$= (3.269 \times 10^{7}\,W/m^2)\,4\pi(6.96 \times 10^{9}\,m)^2$

$= \underline{\underline{1.99 \times 10^{28}}}\,W$

(b) $\lambda_{max} T = 2.898 \times 10^{-3}\,m.K$

$\therefore \lambda_{max} = \dfrac{2.898 \times 10^{-3}\,m.K}{4900K} = \underline{\underline{5.914 \times 10^{-7}}}\,m$

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2. Electron is confined to a space that is, $L = 2.0 \times 10^{-10}$, wide.

Uncertainty principle: $\Delta x \Delta p_x \geq \frac{\hbar}{2}$

$\Delta x \approx L/2$

(a)

$$\therefore \Delta p_x \geq \frac{\hbar}{2\Delta x} = \frac{1.055 \times 10^{-34} \text{ J.s}}{2(1.0 \times 10^{-10})} = 5.275 \times 10^{-25} \text{ kg.m/s}$$

(b) but $\Delta p_x^2 = \langle p_x^2 \rangle - \langle p_x \rangle^2$ and $\langle p_x \rangle = 0$

$$\therefore \Delta p_x^2 = \langle p_x \rangle^2 = 2.783 \times 10^{-49} \text{ kg}^2 \cdot \text{m}^2/\text{s}^2$$

$$E_k = \frac{1}{2m} \langle p_x \rangle^2$$

$$\therefore E_k \geq \frac{1}{2(9.11 \times 10^{-31} \text{ kg})} (2.783 \times 10^{-49} \text{ kg}^2 \cdot \text{m}^2/\text{s}^2)$$

$$\geq 1.527 \times 10^{-19} \text{ kg} \cdot \text{m}^2/\text{s}^2 \text{ or J}$$

$$\geq \frac{1.527 \times 10^{-19} \text{ J}}{1.6 \times 10^{-19} \text{ J/eV}} = 0.9545 \text{ eV}$$

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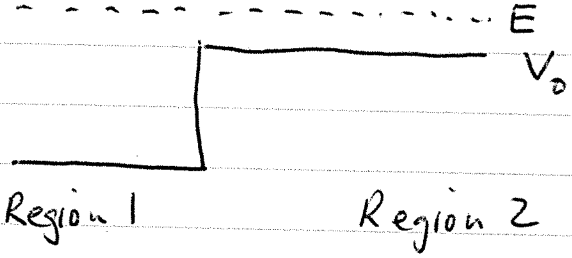
3. Particle incident on a step potential from the left with $E > V_0$




$E > V_0$ so "free particle" like wave functions everywhere

(a) $$k_1 = \sqrt{\frac{2mE}{\hbar^2}}, \quad k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$$

$$k_1 = 3k_2 \quad \therefore \sqrt{\frac{2mE}{\hbar^2}} = 3\sqrt{\frac{2m(E-V_0)}{\hbar^2}}$$

$$\therefore \frac{2mE}{\hbar^2} = 9\left(\frac{2m(E-V_0)}{\hbar^2}\right)$$

$$\therefore E = 9(E-V_0) \Rightarrow 8E = 9V_0$$

$$\therefore E = \left(\frac{9}{8}\right)V_0$$

(b) Most general solutions to the S. Eqn.

Space part:

Region 1 : $\psi_1(x) = Ae^{ik_1x} + Be^{-ik_1x}$ for $x \leq 0$

Region 2 : $\psi_2(x) = Ce^{ik_2x} + De^{-ik_2x}$ for $x \geq 0$

Time part:

$\varphi(t) = e^{-iEt/\hbar}$ for both regions

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3. contd. The complete wave function in each of these regions would be the product of the space part and time part.

The particle is incident from the left therefore there will be no reflected component in Region 2

$\therefore D = 0$

(c) Boundary conditions at $x = 0$

Continuity of $\psi(x)$ :

$$\left. A e^{ik_1 x} + B e^{-ik_1 x} \right|_{x=0} = \left. C e^{ik_2 x} \right|_{x=0}$$

$$\therefore A + B = C \qquad \text{—— (1)}$$

Continuity of $\dfrac{\partial \psi(x)}{\partial x}$ :

$$\left. \frac{\partial}{\partial x}\left(A e^{ik_1 x} + B e^{-ik_1 x}\right) \right|_{x=0} = \left. \frac{\partial}{\partial x}\left(C e^{ik_2 x}\right) \right|_{x=0}$$

$$\therefore k_1 (A - B) = k_2 C \quad \text{or}$$

$$A - B = \frac{k_2}{k_1} C \qquad \text{—— (2)} \quad (3c)$$

3. contd. Adding equations ① and ② gives:

$$2A = \left(1 + \frac{k_2}{k_1}\right)C$$

$$\therefore A = \frac{1}{2}\left(1 + \frac{k_2}{k_1}\right)C \quad \text{but } \frac{k_2}{k_1} = \frac{1}{3}$$

$$\therefore A = \frac{2}{3}C$$

Subtracting ① and ② gives:

$$2B = \left(1 - \frac{k_2}{k_1}\right)C$$

$$\therefore B = \frac{1}{2}\left(1 - \frac{k_2}{k_1}\right)C \quad \text{but } \frac{k_2}{k_1} = \frac{1}{3}$$

$$\therefore B = \frac{1}{3}C$$

Summary:

$$\psi_1(x) = \left(\frac{2}{3}C\right)e^{ik_1x} + \left(\frac{1}{3}C\right)e^{-ik_1x}$$

$$\psi_2(x) = Ce^{ik_2x} \qquad k_1 = 3k_2 = \sqrt{\frac{2mE}{\hbar^2}}$$

$$\varphi(t) = e^{-iEt/\hbar}$$

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**3. contd**

**(d)** Probability densities : $P(x) = \Psi^* \Psi$

Region 1 : $P_1(x) = \left[\left(\frac{2}{3} c^* e^{-ik_1x} + \frac{1}{3} c^* e^{ik_1x}\right)\right.$

$$\left(\frac{2}{3} c e^{ik_1x} + \frac{1}{3} c e^{-ik_1x}\right)$$

$$\therefore P_1(x) = \frac{4}{9} c^* c + \frac{1}{9} c^* c + \frac{2}{9} c^* c \, e^{-i2k_1x} + \frac{2}{9} c^* c e^{+i2k_1x}$$

$$= c^* c \left( \frac{5}{9} + \frac{2}{9} c^* c \left( e^{i2k_1x} + e^{-i2k_1x} \right) \right)$$

using $\cos u = \dfrac{e^{iu} - e^{-iu}}{2}$

$$P_1(x) = c^* c \left( \frac{5}{9} + 2\left(\frac{2}{9}\right) \cos 2k_1x \right)$$

$$\therefore P_1(x) = c^* c \left( \frac{5}{9} + \frac{4}{9} \cos 2k_1x \right)$$

Region 2 : $P_2(x) = \left( c^* e^{-ik_2x} \right)\left( c e^{ik_2x} \right)$

$$= c^* c$$

$\therefore P_2(x) = c^* c$ constant for $x \geq 0$

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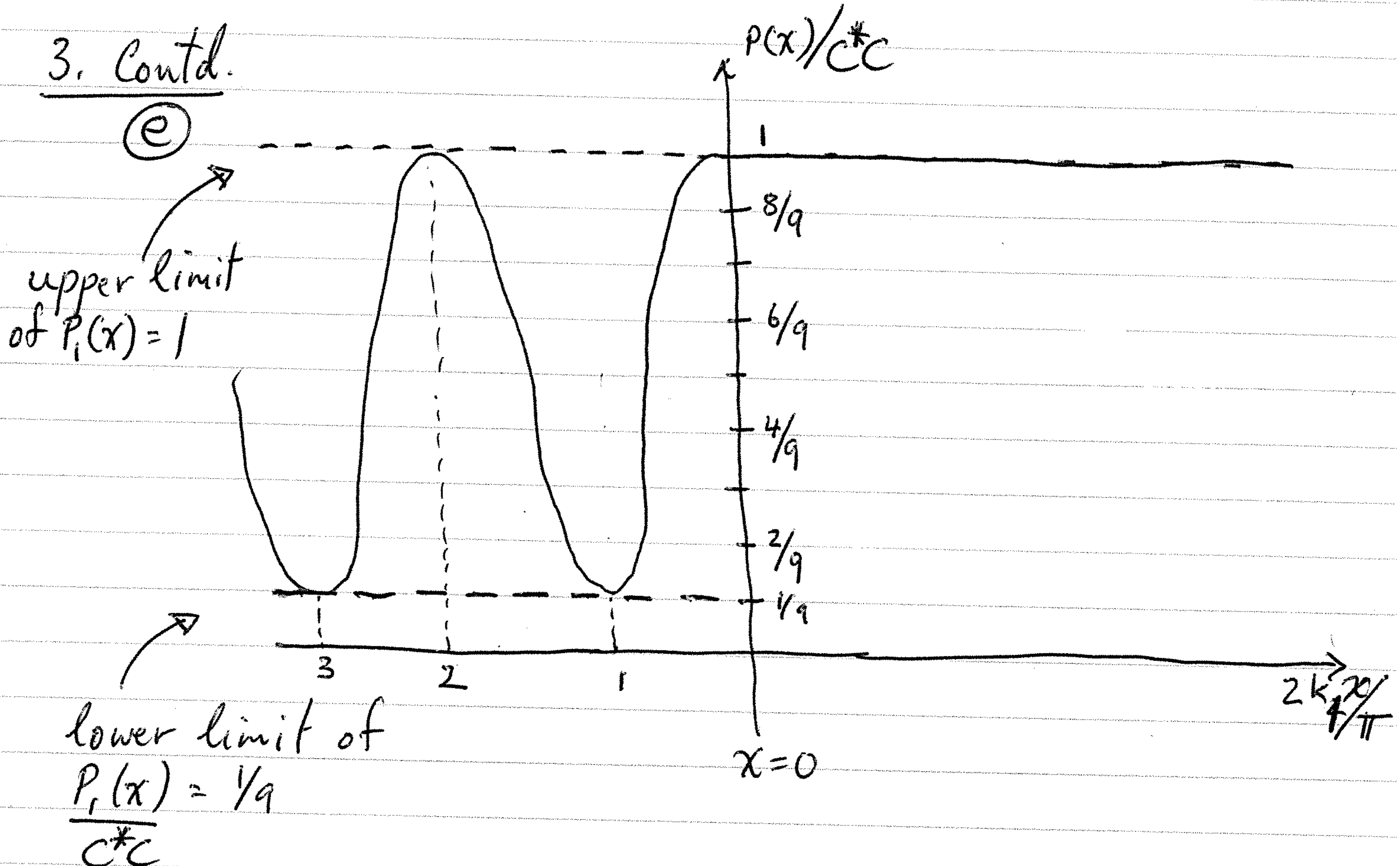
3. Contd.

e




— Probability density is constant in Region 2

— Probability density oscillates between a maxima of 1 and minima of 1/9

— maxima are located at $2k_1x = 2n\pi$ or $k_1x = n\pi$ for $n = 0, 1, 2, \ldots$

— minima are located at $2k_1x = (2n+1)\pi$ for $n = 0, 1, 2 \ldots$ or $k_1x = (n + 1/2)\pi$

— Probabilities are expressed relative to $c^*c$

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④ Particle confined to an infinite square well $0 \le x \le L$

$$\Psi(x,t) = \sqrt{\frac{2}{L}}\,\sin\left(\frac{\pi x}{L}\right)e^{-iEt/\hbar} \qquad 0 \le x \le L$$

$= $ zero everywhere else

ⓐ
$$\langle x\rangle = \int_{-\infty}^{\infty}\Psi^*(x,t)\,x\,\Psi(x,t)\,dx$$

$$= \int_0^L \left(\frac{2}{L}\right) x \sin^2\left(\frac{\pi x}{L}\right) dx$$

let $u = \frac{\pi x}{L}$ then $x = \left(\frac{L}{\pi}\right)u \Rightarrow dx = \left(\frac{L}{\pi}\right)du$

$$\therefore \langle x\rangle = \left(\frac{2}{L}\right)\int_{x=0}^{x=L}\left(\frac{L}{\pi}\right)u\left(\sin^2 u\right)\left(\frac{L}{\pi}\right)du$$

$$= \left(\frac{2L}{\pi^2}\right)\int_{x=0}^{x=L} u \sin^2 du = \left(\frac{2L}{\pi^2}\right)\int_{u=0}^{u=\pi} u\sin^2 u\, du$$

using the integrals provided:

$$\langle x\rangle = \left(\frac{2L}{\pi^2}\right)\left[\frac{u^2}{4} - \frac{u\sin 2u}{4} - \frac{\cos 2u}{8}\right]_{u=0}^{u=\pi}$$

$$= \frac{2L}{\pi^2}\left[\left(\frac{\pi^2}{4} - 0 - \frac{1}{8}\right) - \left(0 - 0 - \frac{1}{8}\right)\right] = \left(\frac{2L}{\pi^2}\right)\left(\frac{\pi^2}{4}\right)$$

$$\therefore \langle x\rangle = L/2$$

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4. contd

(a) $\langle x^2 \rangle = \int_{-\infty}^{\infty} \Psi^* x^2 \Psi \, dx = \int_0^L \left(\frac{2}{L}\right) x^2 \sin^2\!\left(\frac{\pi x}{L}\right) dx$

using the same change of variables as earlier

$$\langle x^2 \rangle = \left(\frac{2}{L}\right) \int_{x=0}^{x=L} \left(\frac{L}{\pi}\right)^2 u^2 \sin^2 u \left(\frac{L}{\pi}\right) du$$

$$= \left(\frac{2}{L}\right)\left(\frac{L^2}{\pi^2}\right)\left(\frac{L}{\pi}\right) \int_0^L u^2 \sin^2 u \, du$$

using the integrals provided

$$\langle x^2 \rangle = \frac{2L^2}{\pi^3} \left[ \frac{u^3}{6} - \left(\frac{u^2}{4} - \frac{1}{8}\right) \sin 2u - \frac{u \cos 2u}{4} \right]_0^{\pi}$$

$$\cancel{= \left(\frac{2L^2}{\pi^3}\right)\left(\frac{\pi^3}{6}\right) = \frac{L^2}{3}}$$

$$= \frac{2L^2}{\pi^3} \left[ \left( \frac{\pi^3}{6} - 0 - \frac{\pi}{4} \right) - \left( 0 - 0 - \cancel{\frac{\pi}{4}}^{\,0} \right) \right]$$

$$= \frac{2L^2}{\pi^3} \left( \frac{\pi^3}{6} - \frac{\pi}{4} \right) = L^2 \left( \frac{1}{3} - \frac{1}{2\pi^2} \right)$$

$= 0.2827\, L^2$

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4. contd

(b) $p_{op} = -i\hbar \frac{\partial}{\partial x}$ or $\frac{\hbar}{i}\frac{\partial}{\partial x}$

$$\therefore \langle p \rangle = \int_{-\infty}^{\infty} \Psi^* p_{op} \Psi \, dx = \int_0^L \psi \, p_{op} \, \psi \, dx$$

$$= \int_0^L \left(\frac{2}{L}\right) \sin\left(\frac{\pi x}{L}\right) \left(\frac{\hbar}{i}\right) \frac{\partial}{\partial x}\left(\sin \frac{\pi x}{L}\right) dx$$

$$= \frac{2\hbar}{iL} \int_0^L \left(\sin \pi x\right) \left(\cos \frac{\pi x}{L}\right) \left(\frac{\pi}{L}\right) dx$$

$$= \left(\frac{2\hbar\pi}{iL^2}\right) \int_0^L \sin \frac{\pi x}{L} \cos \frac{\pi x}{L} \, dx$$

Again transforming from $x$ to $u = \frac{\pi x}{L}$

$$\langle p \rangle = \left(\frac{2\hbar\pi}{iL^2}\right)\left(\frac{L}{\pi}\right) \int_0^{\pi} \sin u \cos u \, du$$

$$= \frac{2\hbar}{iL} \left(\frac{\sin u^2}{2}\right)\Bigg|_{u=0}^{u=\pi} = 0$$

$$\therefore \langle p \rangle = 0$$

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4. contd

(b) $\langle p^2 \rangle = \int_{-\infty}^{\infty} \Psi^* p_{op} p_{op} \Psi \, dx = \int_0^L \Psi(x)^* p_{op} p_{op} \Psi(x) \, dx$

$$= \left(\frac{2}{L}\right) \int_0^L \sin\left(\frac{\pi x}{L}\right)\left(-i\hbar \frac{\partial}{\partial x}\right)\left(-i\hbar \frac{\partial}{\partial x}\right) \sin\left(\frac{\pi x}{L}\right) dx$$

$$-i\hbar \frac{\partial}{\partial x}\left(\sin \frac{\pi x}{L}\right) = -i\hbar \left(\frac{\pi}{L}\right) \cos \frac{\pi x}{L}$$

$$-i\hbar \frac{\partial}{\partial x}\left(-i\hbar\left(\frac{\pi}{L}\right)\cos\frac{\pi x}{L}\right) = (-i\hbar)(-i\hbar)\left(\frac{\pi}{L}\right)\left(\frac{-\pi}{L}\right)\sin\frac{\pi x}{L}$$

$$= \frac{\hbar^2 \pi^2}{L^2} \sin \frac{\pi x}{L}$$

$$\therefore \langle p^2 \rangle = \left(\frac{2}{L}\right) \int_0^L \sin\left(\frac{\pi x}{L}\right) \frac{\hbar^2 \pi^2}{L^2} \sin\left(\frac{\pi x}{L}\right) dx$$

$$= \frac{2\hbar^2 \pi^2}{L^3} \int_0^L \sin^2 \frac{\pi x}{L} \, dx$$

using the same transformation from $x \to u$

$$\langle p^2 \rangle = \left(\frac{2\hbar^2 \pi^2}{L^3}\right)\left(\frac{L}{\pi}\right) \int_0^{\pi} \sin^2 u \, du$$

$$= \frac{2\hbar^2 \pi}{L^2} \left[\frac{1}{2}(u - \sin u \cos u)\right]_{u=0}^{u=\pi}$$

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4. contd.

(b) $$\langle p^2 \rangle = \left(\frac{2\hbar^2\pi}{L^2}\right)\left(\frac{\pi}{2}\right) = \frac{\hbar^2\pi^2}{L^2}$$

(c) $$(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2 = 0.2827L^2 - \left(\frac{L}{2}\right)^2$$

$$= 0.0327\,L^2$$

$$\therefore \Delta x = 0.181\,L$$

$$(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2 = \langle p^2 \rangle - 0 = \langle p^2 \rangle$$

$$= \frac{\hbar^2\pi^2}{L^2}$$

$$\therefore \Delta p = \frac{\hbar\pi}{L}$$

(d) $$\Delta x\,\Delta p = (0.181\,L)\left(\frac{\hbar\pi}{L}\right) = 0.181\,\hbar\pi$$

$$= 0.568\,\hbar$$

The uncertainty principle requires that

$$\Delta x \Delta p \geq \frac{\hbar}{2} = 0.5\,\hbar$$

In our case $\Delta x \Delta p = 0.568\,\hbar$ therefore the uncertainty principle is satisfied