PHYS 2380

PRACTICE PROBLEMS

NOT TO BE SUBMITTED!

① $\psi_1(x) = \left(\frac{4}{\pi}\frac{m^3\omega^3}{\hbar^3}\right)^{1/4} x e^{-m\omega x^2/2\hbar}$

a.) $\langle x^2 \rangle = \int_{-\infty}^{\infty} \psi_1^* x^2 \psi_1 \, dx$

$$= \int_{-\infty}^{\infty} \left(\frac{4}{\pi}\frac{m^3\omega^3}{\hbar^3}\right)^{1/2} x^4 e^{-m\omega x^2/\hbar} dx$$

$$= \left(\frac{4}{\pi}\frac{m^3\omega^3}{\hbar^3}\right)^{1/2} \int_{-\infty}^{\infty} x^4 e^{-m\omega x^2/\hbar} dx$$

Set $u = \sqrt{\frac{m\omega}{\hbar}}\, x = \sqrt{\alpha}\, x$ $\qquad du = \sqrt{\alpha}\, dx, \quad dx = \frac{du}{\sqrt{\alpha}}$

$u^4 = \alpha^2 x^4 \qquad x^4 = \frac{u^4}{\alpha^2}$

$$\langle x^2 \rangle = \left(\frac{4}{\pi}\alpha^3\right)^{1/2} \frac{1}{\alpha^{5/2}} \int_{-\infty}^{\infty} u^4 e^{-u^2} du$$

$$= \left(\frac{4}{\pi}\right)^{1/2} \frac{\alpha^{3/2}}{\alpha^{5/2}} \cdot \frac{3\sqrt{\pi}}{4}$$

$$= \frac{3}{2\alpha}$$

$$\boxed{\langle x^2 \rangle = \frac{3\hbar}{2m\omega}}$$

b.) $\langle p^2 \rangle = \int_{-\infty}^{\infty} \psi_1^* p_{op}^2 \psi_1 \, dx$

$$= \int_{-\infty}^{\infty} \psi_1^* \left(-\hbar^2 \frac{\partial^2 \psi_1}{\partial x^2}\right) dx$$

$$\frac{\partial \psi_1}{\partial x} = x\left(-\frac{m\omega x}{\hbar}\right)e^{-m\omega x^2/2\hbar} + e^{-m\omega x^2/2\hbar}$$

$$\frac{\partial^2 \psi_1}{\partial x^2} = \left(-\frac{m\omega x^2}{\hbar}\right)\left(-\frac{m\omega x}{\hbar}\right)e^{-m\omega x^2/2\hbar} + \left(-\frac{2m\omega}{\hbar}x\right)e^{-m\omega x^2/2\hbar}$$

$$\left(-\frac{m\omega}{\hbar}x\right)e^{-m\omega x^2/2\hbar}$$

$$= \frac{m^2\omega^2 x^3}{\hbar^2}e^{-m\omega x^2/2\hbar} - \frac{3m\omega x}{\hbar}e^{-m\omega x^2/2\hbar}$$

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} -\hbar^2 \left(\left(\frac{4}{\pi}\frac{m^3\omega^3}{\hbar^3}\right)^{1/2} x \, e^{-m\omega x^2/2\hbar}\right)$$

$$\cdot \left(\frac{m^2\omega^2 x^3}{\hbar^2}e^{-m\omega x^2/2\hbar} - \frac{3m\omega x}{\hbar}e^{-m\omega x^2/2\hbar}\right) dx$$

$$= -\hbar\left(\frac{4}{\pi}\frac{m^3\omega^3}{\hbar^3}\right)^{1/2} \int_{-\infty}^{\infty} \left(\frac{m^2\omega^2}{\hbar^2}x^4 e^{-m\omega x^2/\hbar} - \frac{3m\omega x^2}{\hbar}e^{-m\omega x^2/\hbar}\right) dx$$

$$= -\hbar\left(\frac{4}{\pi}\frac{m^3\omega^3}{\hbar^3}\right)^{1/2} \left[\int_{-\infty}^{\infty} \frac{m^2\omega^2}{\hbar^2} x^4 e^{-m\omega x^2/\hbar} dx + \int_{-\infty}^{\infty} \left(\frac{-3m\omega}{\hbar}\right) x^2 e^{-m\omega x^2/\hbar} dx\right]$$

Set $\alpha = \frac{m\omega}{\hbar}$, $u = \sqrt{\alpha}\, x$ $\quad dx = \frac{du}{\sqrt{\alpha}}$

$u^4 = \alpha^2 x^4 = \frac{m^2\omega^2}{\hbar^2}x^4 \qquad u^2 = \alpha x = \frac{m\omega}{\hbar}x^2$

$$\langle p^2\rangle = -\hbar^2\left(\frac{4}{\pi}\frac{m^3\omega^3}{\hbar^3}\right)^{1/2}\frac{1}{\sqrt{\alpha}}\left[\int_{-\infty}^{\infty}u^4 e^{-u^2}du - 3\int_{-\infty}^{\infty}u^2 e^{-u^2}du\right].$$

$$= -\hbar^2\left(\frac{4}{\pi}\right)^{1/2}\frac{\alpha^{3/2}}{\alpha^{1/2}}\left[\frac{3\sqrt{\pi}}{4} - \frac{6\sqrt{\pi}}{4}\right]$$

From the identities

$$\int_{-\infty}^{\infty} u^4 e^{-u^2}du = \frac{3\sqrt{\pi}}{4}, \quad \int_{-\infty}^{\infty} u^2 e^{-u^2}du = \frac{\sqrt{\pi}}{2}$$

$$\langle p^2\rangle = -\hbar^2\left(\frac{4}{\pi}\right)^{1/2}\alpha^{2/2}\left[-\frac{3\sqrt{\pi}}{4}\right]$$

$$= \frac{3\hbar^2\alpha}{2}$$

$$= \frac{3\hbar^2}{2}\frac{m\omega}{\hbar}$$

$$\boxed{\langle p^2\rangle = \frac{3m\hbar\omega}{2}}$$

c.) if $\langle p\rangle = 0$ and $\langle x\rangle = 0$

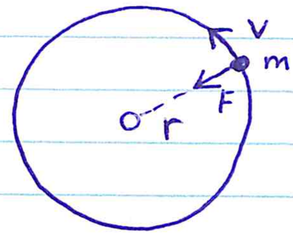
$$\Delta p = \sqrt{\langle p^2\rangle} \quad \text{and} \quad \Delta x = \sqrt{\langle x^2\rangle}$$

$$= \sqrt{\frac{3}{2}\hbar m\omega} \qquad = \sqrt{\frac{3\hbar}{2}\frac{1}{m\omega}}$$

$$\boxed{\Delta p\,\Delta x = \frac{3}{2}\hbar > \frac{\hbar}{2}}$$

②

a.) $F(r) = -\frac{dV(r)}{dr}$, $V(r) = Cr^2$



$F(r) = -2Cr$

This is the direction in which the centripetal force acts.

$$F_c = F$$
$$\frac{mv^2}{r} = 2Cr$$

$$T = \frac{mv^2}{2} = Cr^2$$

Total energy

$T + V = E$

$Cr^2 + Cr^2 = 2Cr^2$

$$\boxed{E = 2Cr^2}$$

b.) $L = pr$

$\therefore\ p = \frac{L}{r}$ $\qquad T = \frac{p^2}{2m} = \frac{L^2}{2mr^2}$

$$Cr^2 = \frac{L^2}{2mr^2}$$

$$r^4 = \frac{L^2}{2mC} = \frac{n^2\hbar^2}{2mC}$$

$$r^2 = \frac{n\hbar}{\sqrt{2mC}}$$

$$E = 2Cr^2 = 2C\frac{n\hbar}{\sqrt{2mC}} = n\hbar\sqrt{\frac{2C}{m}}$$

$$\boxed{E = n\hbar\sqrt{\frac{2C}{m}}}$$

3a) $P(x) = \psi^* \psi$ $\qquad \psi(x) = A \sin\left(\frac{\pi x}{a}\right)$

$$P(x) = A^2 \sin^2\left(\frac{\pi x}{a}\right)$$

$$\int_{-\infty}^{\infty} P(x)\,dx = A^2 \int_0^a \sin^2\left(\frac{\pi x}{a}\right) dx = 1$$

Identity: $\sin^2\alpha = \frac{1 - \cos 2\alpha}{2}$

$$1 = \frac{A^2}{2} \int_0^a \left[1 - \cos\left(\frac{2\pi x}{a}\right)\right] dx$$

$$1 = \frac{A^2}{2}\left(a - \int_0^a \cos\left(\frac{2\pi x}{a}\right) dx\right) \qquad u = \frac{2\pi x}{a},\ \frac{a\,du}{2\pi} = dx.$$

$$1 = \frac{A^2}{2}\left(a - \frac{a}{2\pi}\int_0^{2\pi} \cos u\,du\right)$$

$$1 = \frac{A^2}{2}\left(a - \frac{a}{2\pi}\left(+\sin u\right)_0^{2\pi}\right)$$

$$1 = A^2 \frac{a}{2}$$

$$\frac{2}{a} = A^2$$

$$\boxed{A = \sqrt{\frac{2}{a}}}$$

$$P = \frac{A^2}{2}\int_0^{a/4}\left[1-\cos\left(\frac{2\pi x}{a}\right)\right]dx \qquad u = \frac{2\pi x}{a}, \quad du = \frac{2\pi}{a}dx$$

$$\frac{2\pi}{a}\cdot\frac{a}{4} = \frac{\pi}{2}$$

$$= \frac{A^2}{2}\left[\frac{a}{4} - \frac{a}{2\pi}\int_0^{\pi/2}\cos u\,du\right]$$

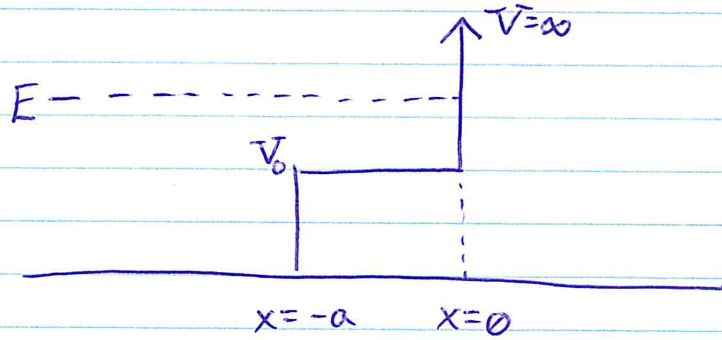
$$= \frac{A^2}{2}\left[\frac{a}{4} - \frac{a}{2\pi}\left(+\sin u\right)_0^{\pi/2}\right]$$

$$= \frac{A^2}{2}\left[\frac{a}{4} - \frac{a}{2\pi}\right] \qquad A^2 = \frac{2}{a} \text{ from previous}$$

$$= \frac{1}{a}\left[\frac{a}{4} - \frac{a}{2\pi}\right] = \frac{1}{4} - \frac{1}{2\pi} = \frac{2\pi}{8\pi} - \frac{4}{8\pi}$$

$$\boxed{P = \frac{2\pi - 4}{8\pi} \approx 0.091 = 9.1\%}$$

④



a.) Solutions for the wavefunction

$\psi_1 = Ae^{ik_1x} + Be^{-ik_1x}$ $\quad x < -a$

$\psi_2 = Ce^{ik_2x} + De^{-ik_2x}$ $\quad \emptyset > x > -a$

$\psi_3 = \emptyset$ $\quad x > \emptyset$

$k_1 = \frac{\sqrt{2mE}}{\hbar}$

$k_2 = \frac{\sqrt{2m(E-V_0)}}{\hbar}$

k_3 is undefined. There is no amplitude in region 3.

Boundary conditions at $x=\emptyset$:

$C + D = \emptyset$

Derivative is discontinuous — No constraint
ie, we must have a node at $x=\emptyset$.

$\therefore \; C = -D$

$\therefore \; \psi_2(x) = Ce^{ik_2x} - Ce^{-ik_2x}$

$= C(2i)\sin k_2x$

$= C'\sin k_2x.$ with $C' = 2iC$

at $x=-a$,

$$Ae^{-ik_1a} + Be^{ik_1a} = C\sin(-k_2a) = -C\sin k_2a.$$

Derivative is finite here, and gives:

$$ik_1\left(Ae^{-ik_1a} - Be^{ik_1a}\right) = Ck_2\cos(-k_2a) = Ck_2\cos(k_2a).$$

$$Ae^{-ik_1a} - Be^{ik_1a} = \frac{Ck_2}{ik_1}\cos k_2a$$

$$2Ae^{-ik_1a} = -C\sin k_2a + \frac{Ck_2}{ik_1}\cos k_2a$$

$$\boxed{A = -\frac{C}{2}e^{ik_1a}\left[\sin k_2a + \frac{ik_2}{k_1}\cos k_2a\right]}$$

similarly

$$2Be^{ik_1a} = -C\sin(k_2a) - \frac{Ck_2}{ik_1}\cos k_2a$$

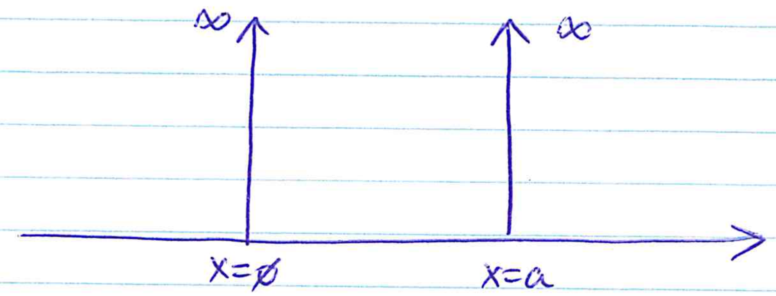
$$\therefore \boxed{B = -\frac{C}{2}e^{-ik_1a}\left[\sin k_2a - \frac{ik_2}{k_1}\cos k_2a\right]}$$

c.) $B = A^*$ above

$$|B|^2 = |A|^2 \qquad R = \frac{|B|^2}{|A|^2} = 1$$

reflection coefficient

⑤ ψ only exists in the region $0 \leq x \leq a$



a.) ψ is a solution to $-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = E\psi$

$\therefore$ ψ has the form

$$\psi(x) = Ae^{ikx} + Be^{-ikx}$$

Applying the boundary conditions gives

$x=0$: $\quad A+B=0$

$$\therefore \quad A=-B$$

$$\therefore \quad \psi(x) = Ae^{ikx} - Ae^{-ikx}$$

$$= A'\sin(kx) \qquad A' = 2iA.$$

Apply BCs at $x=a$:

$$A\sin ka = 0$$

$$\therefore \quad ka = n\pi \quad \text{where } n=1,2,3\ldots$$

$$k = \frac{n\pi}{a}.$$

Momentum

$p=\hbar k$

So Kinetic energy $T = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}$

$$= \frac{\hbar^2 \pi^2 n^2}{2ma^2}$$

<u>Normalization:</u>

$A = \sqrt{\frac{2}{a}}$ from an earlier problem!

b) $\langle x \rangle = \int_0^a \psi^* x \psi \, dx$

$$= \int_0^a \frac{2}{a} \sin\left(\frac{n\pi x}{a}\right) \cdot x \cdot \sin\left(\frac{n\pi x}{a}\right) dx$$

$$= \frac{2}{a} \int_0^a x \sin^2\left(\frac{n\pi x}{a}\right) dx$$

$u = \frac{n\pi x}{a} \qquad dx = \frac{a}{n\pi} du$

$x = \frac{a}{n\pi} u.$

$$\langle x \rangle = \frac{2}{a} \int_0^{n\pi} \left(\frac{a}{n\pi}\right)^2 u \sin^2 u \, du$$

$$= \frac{2a}{(n\pi)^2} \int_0^{n\pi} u \sin^2 u \, du$$

$$= \frac{2a}{(n\pi)^2} \left[\frac{u^2}{4} - \frac{u \sin 2u}{4} - \frac{\cos 2u}{8} \right]_0^{n\pi}$$

$$= \frac{2a}{(n\pi)^2}\left[\frac{(n\pi)^2}{4} - \frac{1}{8} + \frac{1}{8}\right] = \frac{a}{2}$$

$$\langle x^2 \rangle = \int_0^a \psi^* x^2 \psi \, dx$$

$$= \frac{2}{a}\int_0^a x^2 \sin^2\left(\frac{n\pi x}{a}\right) dx$$

Transforming in the usual manner

$$\langle x^2 \rangle = \frac{2}{a}\int_0^{n\pi} \left(\frac{a}{n\pi}\right)^2 u^2 \sin^2 u \left(\frac{a}{n\pi}\right) du$$

$$= \frac{2}{a}\left(\frac{a}{n\pi}\right)^3 \int_0^{n\pi} u^2 \sin^2 u \, du$$

$$= \frac{2a^2}{(n\pi)^3}\left(\frac{u^3}{6} - \left(\frac{u^2}{4} - \frac{1}{8}\right)\sin 2u - \frac{u\cos 2u}{4}\right)_0^{n\pi}$$

$$= \frac{2a^2}{(n\pi)^3}\left(\frac{(n\pi)^3}{6} - \frac{n\pi}{4}\right)$$

$$\langle x^2 \rangle = 2a^2\left(\frac{1}{6} - \frac{1}{4n^2\pi^2}\right)$$

$$(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2$$

$$= 2a^2\left[\frac{1}{6} - \frac{1}{4n^2\pi^2}\right] - \frac{a^2}{4}$$

$$= \frac{a^2}{3} - \frac{a^2}{2n^2\pi^2} - \frac{a^2}{4} \Rightarrow \Delta x = a\left(\frac{1}{12} - \frac{1}{2\pi^2 n^2}\right)^{1/2}$$

(6) a.) $\langle r_{20} \rangle = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \psi_{n\ell m}^* \, r \, \psi_{n\ell m} \, r^2 \sin\theta \, dr \, d\theta \, d\phi$

$$= \iiint (R_{20} \Theta_{00} \Phi_0)^* \, r \, R_{20} \Theta_{00} \Phi_0 \, r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \int_0^{\infty} r^3 R_{20}^* R_{20} \, dr \int_0^{2\pi}\int_0^{\pi} (\Theta \Phi)^* \Theta_{00} \Phi_0 \sin\theta \, d\theta \, d\phi$$

$\Theta_{00} \Phi_0 = Y_{00}(\theta, \phi)$ is normalized, so the integral over θ and ϕ is $\underline{1}$.

This reduces to

$$\langle r_{20} \rangle = \int_0^{\infty} r^3 R_{20}^* R_{20} \, dr$$

$$= \int_0^{\infty} r^3 \left[\frac{1}{\sqrt{2a_0^3}} \left(1 - \frac{r}{2a_0}\right) e^{-r/2a_0} \right]^2 dr$$

$$= \frac{1}{2a_0^3} \int_0^{\infty} r^3 \left(1 - \frac{r}{2a_0}\right)^2 e^{-r/a_0} \, dr$$

$$= \frac{1}{2a_0^3} \int_0^{\infty} r^3 \left(1 - \frac{r}{a_0} + \frac{r^2}{4a_0^2}\right) e^{-r/a_0} \, dr$$

$$= \frac{1}{2a_0^3} \left[\int_0^{\infty} r^3 e^{-r/a_0} dr - \int_0^{\infty} \frac{r^4}{a_0} e^{-r/a_0} dr + \frac{1}{4} \int_0^{\infty} \frac{r^5}{a_0^2} e^{-r/a_0} dr \right]$$

$$= \frac{1}{2} \left[\int_0^{\infty} \frac{r^3}{a_0^3} e^{-r/a_0} dr - \int_0^{\infty} \frac{r^4}{a_0^4} e^{-r/a_0} dr + \frac{1}{4} \int_0^{\infty} \frac{r^5}{a_0^5} e^{-r/a_0} dr \right]$$

Set $u = \frac{r}{a_0}$, $a_0 \, du = dr$

$$\langle r_{20}\rangle = \frac{a_0}{2}\left[\int_0^\infty u^3 e^{-u}\,du - \int_0^\infty u^4 e^{-u}\,du + \frac{1}{4}\int_0^\infty u^5 e^{-u}\,du\right]$$

a_0 factor from du transformation

$$a_0\,du = dr$$

Use $\int_0^\infty u^n e^{-u}\,du = n!$

$$\langle r_{20}\rangle = \frac{a_0}{2}\left[3! - 4! + \frac{5!}{4}\right]$$

$$= \frac{a_0}{2}\left[6 - 24 + 30\right]$$

$$= \frac{12}{2}a_0$$

$$\boxed{\langle r_{20}\rangle = 6a_0}$$

For the Bohr model the predicted radius is

$$r_{Bohr} = \frac{n^2 a_0}{Z} = 4a_0$$

QM result is $1.5 \times r_{Bohr}$

b.) Is R_{21} normalized?

$$P = \int_0^\infty R_{21}^* R_{21} r^2 dr \underbrace{\int_0^{2\pi}\int_0^{\pi} Y_{2m}(\theta,\phi) \sin\theta \, d\theta \, d\phi}_{\text{Assume normalized}}$$

$$P = \int_0^\infty R_{21}^* R_{21} r^2 dr$$

$$= \int_0^\infty \left[\frac{1}{2\sqrt{6a_0^3}} \left(\frac{r}{a_0}\right) e^{-r/2a_0} \right]^2 r^2 dr$$

$$= \frac{1}{24a_0^3} \int_0^\infty \frac{r^4}{a_0^2} e^{-r/a_0} dr \qquad \frac{r}{a_0} = u, \; a_0 du = dx$$

$$= \frac{a_0}{24a_0} \int_0^\infty u^4 e^{-u} du = \frac{a_0}{24a_0} \left[4! \right] = \frac{1}{24} \cdot 24$$

$$\boxed{P = 1}$$

R_{21} is Normalized!

c.) Is $Y_{11} = \Theta_{11}\Phi_1$ normalized by itself?

Evaluate: $P = \int_0^{2\pi}\int_0^{\pi} Y_{11}^* Y_{11} \sin\theta \, d\theta \, d\phi$

$$= \int_0^{2\pi}\int_0^{\pi} \left(\sqrt{\frac{3}{8\pi}} \sin\theta e^{-i\phi}\right)\left(\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi}\right) \sin\theta \, d\theta \, d\phi$$

$$= \frac{3}{8\pi} \int_0^{\pi} \sin^3\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= \frac{3}{4} \int_0^{\pi} \sin^3\theta \, d\theta = \frac{3}{4}\left[\frac{-1}{3}\cos\theta\left(\sin^2\theta + 2\right)\right]_0^{\pi}$$

$$= \frac{3}{4}\left[\frac{2}{3} + \frac{2}{3}\right] = 1 \qquad \boxed{P=1}$$

$\therefore$ $Y_{11} = \Theta_{11}\Phi_1$ is normalized

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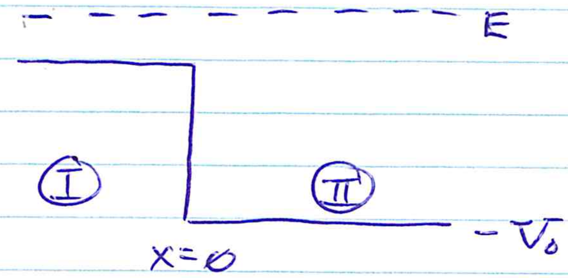


Region I: $x < 0$

$$\psi_1(x) = Ae^{ik_1x} + Be^{-ik_1x}$$

Region II: $x > 0$

$$\psi_2(x) = Ce^{ik_2x} + \cancel{De^{-ik_2x}}$$

Only a transmitted wave in(

b.)

$$k_1 = \frac{\sqrt{2mE}}{\hbar}$$

If $V_0 = 3E$

$\downarrow$

$$k_2 = \frac{\sqrt{2m(E-V)}}{\hbar} = \frac{\sqrt{2m(E+V_0)}}{\hbar} = \frac{\sqrt{2m(4E)}}{\hbar} = \frac{2\sqrt{2mE}}{\hbar} = 2k_1$$

c.) Apply BCs:

$x=0$: $\quad A + B = C \quad$ (1)

$$ik_1A - ik_1B = ik_2C$$

$$\therefore \quad A - B = \frac{k_2}{k_1}C = 2C \quad (2)$$

$$(1) + (2) \Rightarrow \quad 2A = 3C \quad \text{or} \quad A = \frac{3}{2}C$$

$$(1) - (2) \Rightarrow \quad 2B = C - 2C$$

$$2B = -C$$

$$B = -\frac{1}{2}C$$

$$\therefore \quad \psi_1(x) = \frac{3}{2}Ce^{ik_1x} - \frac{1}{2}Ce^{-ik_1x}$$

$$\psi_2(x) = Ce^{ik_2x} = Ce^{2ik_1x}$$

d.) <u>Probability density</u>

Region I: $P_I(x) = \psi_1^* \psi_1 = (A^* e^{-ik_1x} + B^* e^{ik_1x})(A e^{ik_1x} + B e^{-ik_1x})$

$$P_I(x) = A^*A + A^*B e^{-2ik_1x} + B^*A e^{2ik_1x} + B^*B$$

$$= (A^*A + B^*B) + 2A^*B \cos(2k_1x)$$

$$= c^*c\left[\frac{9}{4} + \frac{1}{4} - \frac{3}{2}\cos 2k_1x\right]$$

$$P_I(x) = c^*c\left[\frac{5}{2} - \frac{3}{2}\cos 2k_1x\right]$$

Region II: $P_{II}(x) = \psi_2^* \psi_2$

$$= (c^* e^{-ik_2x})(c e^{ik_2x}) = c^*c$$

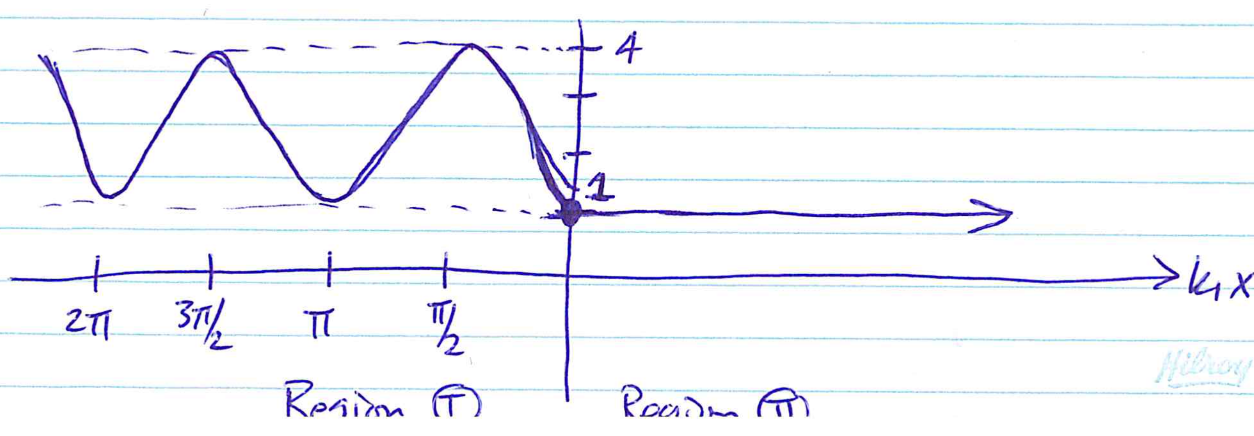
e.) Arbitrarily set $c^*c = 1$ for convenience.

minimum when $2k_1x = 2n\pi$ (argument of
$k_1x = n\pi$ cosine term is even multiple of π (or $\emptyset$).)
$n = \emptyset, 1, 2$

Maximum when argument of cosine term odd multiple of π

$2k_1x = (2n+1)\pi$

$k_1x = n\pi + \pi/2$



f.) Transmission coefficient:

$$T = \frac{k_2}{k_1}\frac{C^*C}{A^*A} = 2\frac{C^*C}{\frac{9}{4}C^*C} = 2\left(\frac{4}{9}\right) = \frac{8}{9}$$

g.) Reflection coefficient:

$$R = \frac{k_1}{k_1}\frac{B^*B}{A^*A} = \frac{\frac{1}{4}C^*C}{\frac{9}{4}C^*C} = \frac{1}{9}$$

h.) $T + R = 1$ QED.

⑧ a.) $\lambda = \frac{h}{p}$ Consider all energies – large and small! This means use relativistic energy expression.

$$E^2 = p^2c^2 + m^2c^4$$

$$p^2c^2 = E^2 - m^2c^4 \qquad E = E_k + mc^2 = \text{kinetic + rest energy}$$

$$p^2c^2 = (E_k + mc^2)^2 - m^2c^4$$

$$= E_k^2 + 2E_kmc^2 + \cancel{m^2c^4} - \cancel{m^2c^4}$$

$$p^2c^2 = E_k^2 + 2E_kmc^2 = E_k^2\left(1 + \frac{2mc^2}{E_k}\right)$$

$$p = \frac{1}{c}(E_k)\left(1 + \frac{2mc^2}{E_k}\right)^{1/2}$$

$$\therefore \quad \lambda = \frac{h}{p} = \frac{hc}{E_k\left(1 + \frac{2mc^2}{E_k}\right)^{1/2}}$$

b.) In the extreme relativistic limit $E_k >> mc^2$

$$\frac{2mc^2}{E_k} << 1 \quad \text{so} \quad 1 + \frac{2mc^2}{E_k} \approx 1$$

Then $\lambda = \frac{hc}{E_k} = \frac{hc}{E}$

In the classical limit $mc^2 >> E_k$

$$1 + \frac{2mc^2}{E_k} \approx \frac{2mc^2}{E_k} \qquad \lambda = \frac{hc}{E_k\sqrt{\frac{2mc^2}{E_k}}} = \frac{h}{E_k\sqrt{\frac{2m}{E_k}}}$$

$\lambda = \frac{h}{\sqrt{2mE_k}}$ Classically $E_k = p^2/2m$ so $\lambda = h$

(1) a.) $u(\nu)d\nu = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/kT}-1}d\nu$

$\nu = \frac{c}{\lambda}$ $\quad d\nu = -\frac{c}{\lambda^2}d\lambda$ ie, ν decreases as λ increases, Interested in the physical behaviour of the distribution though. This requires $d\nu = \left|\frac{c}{\lambda^2}\right| d\lambda$

$$u(\lambda)\,d\lambda = \frac{8\pi h}{c^3}\left(\frac{c}{\lambda}\right)^3\frac{1}{e^{hc/\lambda k_bT}-1}\left(\frac{c}{\lambda^2}\right)d\lambda$$

$$= \frac{8\pi hc}{\lambda^5}\frac{1}{(e^{hc/\lambda k_bT}-1)}d\lambda$$

b.) $R(\nu) = \frac{c}{4}u(\nu) = \frac{2\pi h\nu^3}{c^2}\left(\frac{1}{e^{h\nu/k_bT}-1}\right)$

$$\therefore R_T = \int_0^\infty R(\nu)d\nu = \frac{2\pi h}{c^2}\int_0^\infty \frac{\nu^3}{e^{h\nu/k_bT}-1}d\nu$$

change variables $z = \frac{h\nu}{k_bT}$ $\quad dz = \frac{h}{k_bT}d\nu$

$$d\nu = \frac{k_bT}{h}dz, \quad \nu = \frac{k_bT}{h}z$$

$$R_T = \frac{2\pi h}{c^2}\int_0^\infty\left(\frac{k_bTz}{h}\right)^3\frac{1}{e^z-1}\left(\frac{k_bT}{h}\right)dz$$

$$= \frac{2\pi h}{c^2}\left(\frac{k_bT}{h}\right)^3\left(\frac{k_bT}{h}\right)\int_0^\infty\frac{z^3}{e^z-1}dz = \frac{2\pi h}{c^2}\frac{k_b^4T^4}{h^4}\frac{\pi^4}{15}$$

$$= \frac{2\pi^5k_b^4}{15c^2h^3}T^4 = \sigma T^4 \quad \text{where } \sigma = \frac{2\pi^5k_b^4}{15c^2h^3}$$