PHYS 2380

FINAL EXAM

SOLUTIONS

2017

① $F(r) = -\frac{dV(r)}{dr}$ $\quad V(r) = Cr^2$

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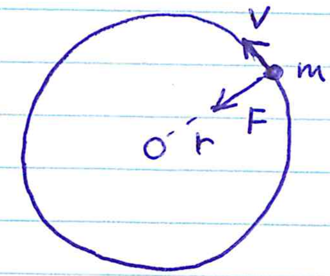

$$F(r) = -2Cr$$

F(r) points toward the center, the direction in which the centripetal force acts

$$F_c = F$$

$$\frac{mv^2}{r} = 2Cr$$

$$T = \frac{mv^2}{2} = Cr^2$$

Total energy $E = T + V$

$$E = Cr^2 + Cr^2 = 2Cr^2$$

$$\boxed{E = 2Cr^2}$$

b.) $L = pr \quad p = \frac{L}{r} \quad T = \frac{p^2}{2m} = \frac{L^2}{2mr^2}$

$$T = \frac{L^2}{2mr^2} = Cr^2$$

$$r^4 = \frac{L^2}{2mC} = \frac{n^2\hbar^2}{2mC}$$

$$r^2 = \frac{n\hbar}{\sqrt{2mC}}$$

$$E = 2Cr^2 = 2C\frac{n\hbar}{\sqrt{2mC}}$$

$$\boxed{E = n\hbar\sqrt{\frac{2C}{m}}}$$

② $\psi(x) = Ae^{-x^2/2a^2}$ $\qquad E = \frac{\hbar^2}{2ma^2}$

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$$\frac{d\psi}{dx} = Ae^{-x^2/2a^2}\left(\frac{-2x}{2a^2}\right) = -\frac{Ax}{a^2}e^{-x^2/2a^2} = -\frac{x}{a^2}\psi$$

$$\frac{d^2\psi}{dx^2} = -\frac{x}{a^2}\frac{d\psi}{dx} + \psi\left(\frac{-1}{a^2}\right) = -\frac{x}{a^2}\left(\frac{-x}{a^2}\psi\right) - \frac{\psi}{a^2}$$

$$= \frac{x^2}{a^4}\psi - \frac{\psi}{a^2} = \frac{1}{a^2}\left(\frac{x^2}{a^2} - 1\right)\psi$$

$$\frac{d^2\psi}{dx^2} = \frac{1}{a^2}\left(\frac{x^2}{a^2} - 1\right)\psi$$

Schrödinger equation: $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$

Re-arrange: $V(x)\psi = E\psi + \frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}$

$$V(x)\psi = \frac{\hbar^2\psi}{2ma^2} + \frac{\hbar^2}{2m}\left(\frac{1}{a^2}\left(\frac{x^2}{a^2} - 1\right)\psi\right)$$

$$= \frac{\hbar^2\psi}{2ma^2} + \frac{\hbar^2 x^2\psi}{2ma^4} - \frac{\hbar^2\psi}{2ma^2}$$

$$\boxed{V(x) = \frac{\hbar^2 x^2}{2ma^4}}$$

b.) This is a simple harmonic oscillator potential! The spring constant is

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$V(x) = \frac{1}{2}Cx^2$, so $\boxed{C = \frac{\hbar^2}{ma^4}}$

③ $\Psi_{n\ell m}(r,\theta,\phi) = R_{n\ell}(r)\,\Theta_{\ell m}(\theta)\,\Phi_m(\phi)$

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a.) $\frac{1}{\Phi_m}\frac{d^2\Phi_m}{d\phi^2} = -m^2$

$\frac{d^2\Phi_m}{d\phi^2} = -m^2\Phi_m$ or $\frac{d^2\Phi_m}{d\phi^2} + m^2\Phi_m = \emptyset.$

$(d^2 + m^2)\Phi_m = \emptyset$ means there are 2 solutions $d = \pm im$

$\Phi_m(\phi) = Ae^{im\phi} + Be^{-im\phi}$

No reflected cmpt, so $B \to \emptyset.$

$L_{zop}\Phi_m(\phi) = -i\hbar\frac{\partial}{\partial\phi}\Phi_m = -i\hbar\frac{d\Phi_m}{d\phi}$

$= -i\hbar\left(Aim\,e^{im\phi}\right) = -i^2 m\hbar\,Ae^{im\phi}$

$\boxed{L_{zop}\Phi_m(\phi) = m\hbar\,\Phi_m(\phi)}$ This is an eigenfcn with eigen value $\boxed{m\hbar.}$

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b.) $\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta_{\ell m}}{d\theta}\right) - \frac{m^2}{\sin^2\theta}\Theta_{\ell m} = -\ell(\ell+1)\Theta_{\ell m}$

$L^2_{op}\Psi_{n\ell m} = -\hbar^2\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta_{\ell m}}{d\theta}R_{n\ell}\Phi_m\right) + \frac{1}{\sin^2\theta}R_{n\ell}\Theta_{\ell m}\frac{\partial^2\Phi_m}{\partial\phi^2}\right]$

$L^2_{op}\Psi_{n\ell m} = -\hbar^2\left[\frac{R_{n\ell}\Phi_m}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{R_{n\ell}\Theta_{\ell m}}{\sin^2\theta}\left(i^2 m\,Ae^{im\phi}\right)\right]$

Nilloy

$Ae^{im\phi} = \Phi_m(\phi)$, so we have

$$L_{op}^2 \psi_{n\ell m} = -\hbar^2\left[\frac{R_{n\ell}\Phi_m}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta_{\ell m}}{d\theta}\right) + \frac{R_{n\ell}\Theta_{\ell m}\Phi_m(-m^2)}{\sin^2\theta}\right]$$

$$= -\hbar^2 R_{n\ell}\Phi_m\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta_{\ell m}}{d\theta}\right) - \frac{m^2\Theta_{\ell m}}{\sin^2\theta}\right]$$

By the $\Theta_{\ell m}$ differential equation, this is

$$L_{op}^2 \psi_{n\ell m} = -\hbar^2 R_{n\ell}\Phi_m\left[-\ell(\ell+1)\Theta_{\ell m}\right]$$

$$= \ell(\ell+1)\hbar^2 R_{n\ell}\Theta_{\ell m}\Phi_m$$

$$\boxed{L_{op}^2 \psi_{n\ell m} = \ell(\ell+1)\hbar^2 \psi_{n\ell m}}$$

This is an eigenfcn eqn with eigenvalue $\boxed{\ell(\ell+1)\hbar^2}$

④ a.) Region 1 : $x<\emptyset$: $\psi_1(x) = Ae^{ik_1x} + Be^{-ik_1x}$

3 Region 2 : $\emptyset \leq x \leq a$: $\psi_2(x) = Ce^{k_2x} + De^{-k_2x}$

Region 3 : $x>a$: $\psi_3(x) = Fe^{ik_3x} + Ge^{-ik_3x}$

b.) $k_1 = \dfrac{\sqrt{2m(E-V)}}{\hbar} = \dfrac{\sqrt{2mV_0}}{\hbar}$

$k_2 = \dfrac{\sqrt{2m(V-E)}}{\hbar} = \dfrac{\sqrt{8mV_0}}{\hbar} = 2k_1$

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$k_3 = \dfrac{\sqrt{2mE}}{\hbar} = \dfrac{\sqrt{18mV_0}}{\hbar} = 3k_1$

c.) There is no reflected wave in region 3 because there is no boundary to reflect off of. G = Ø here. F is the wave travelling to the right in region 3.

2 A is a forward propagating wave (+x) and B is reflected in region 1.

There are no <u>propagating waves</u> in region 2!

d.) The time part is the same in all 3 regions since the particle energy is constant:

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$$\psi_t = \phi(t) = e^{-iEt/\hbar}.$$

e.) Boundary conditions: $x = \emptyset$: $A + B = C + D$ ①

$ik_1A - ik_1B = 2k_1C - 2k_1D$

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② $A - B = \dfrac{2k_1}{ik_1}(C-D) = -2i(C-D)$

At $x=a$: $Ce^{2k_1a} + De^{-2k_1a} = Fe^{i3k_1a}$ ③

$$2k_1Ce^{2k_1a} - 2k_1De^{-2k_1a} = 3ik_1Fe^{i3k_1a}$$

$$Ce^{2k_1a} - De^{-2k_1a} = \frac{3ik_1Fe^{i3k_1a}}{2k_1}$$

$$Ce^{2k_1a} - De^{-2k_1a} = \frac{3i}{2}Fe^{3ik_1a} \quad ④$$

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f.) Add ① + ②: $2A = C + D - 2iC + 2iD$

$$\boxed{A = \frac{C}{2}(1-2i) + \frac{D}{2}(1+2i)}$$

① − ②: $2B = C + D + 2iC - 2iD$

$$\boxed{B = \frac{C}{2}(1+2i) + \frac{D}{2}(1-2i)}$$

③ + ④: $2Ce^{2k_1a} = Fe^{3ik_1a}\left(1 + \frac{3i}{2}\right)$

$$\boxed{C = \frac{F}{2}e^{k_1a(-2+3i)}\left(1 + \frac{3i}{2}\right)}$$

③ − ④: $2De^{-2k_1a} = Fe^{3ik_1a}\left(1 - \frac{3i}{2}\right)$

$$\boxed{D = \frac{F}{2}e^{k_1a(2+3i)}\left(1 - \frac{3i}{2}\right)}$$

In terms of the wave in region 3, we have:

$$A = \frac{1}{2}\left(\frac{F}{2}e^{k_1a(-2+3i)}\left(1+\frac{3i}{2}\right)\right)(1-2i) + \frac{1}{2}\left(\frac{F}{2}e^{k_1a(2+3i)}\left(1-\frac{3i}{2}\right)\right)$$

$$\times (1+2i)$$

$$A = \frac{F}{4}\left[e^{k_1a(-2+3i)}\left(1-2i+\frac{3i}{2}-3i^2\right)\right.$$

$$\left.+ e^{k_1a(2+3i)}\left(1-\frac{3i}{2}+2i-3i^2\right)\right]$$

$$\boxed{A = \frac{F}{4}e^{i3k_1a}\left[e^{-2k_1a}\left(4-\frac{i}{2}\right) + e^{2k_1a}\left(4+\frac{i}{2}\right)\right]}$$

$$B = \frac{1}{2}\left(\frac{F}{2}e^{k_1a(-2+3i)}\left(1+\frac{3i}{2}\right)(1+2i)\right) + \frac{1}{2}\left(\frac{F}{2}e^{k_1a(2+3i)}\left(1-\frac{3i}{2}\right)(1-2i)\right)$$

$$= \frac{F}{4}e^{3ik_1a}\left[e^{-2k_1a}\left(1+2i+\frac{3i}{2}+3i^2\right)\right.$$

$$\left.+ e^{2k_1a}\left(1-2i-\frac{3i}{2}+3i^2\right)\right]$$

$$\boxed{B = \frac{F}{4}e^{3ik_1a}\left[e^{-2k_1a}\left(-2+\frac{7i}{2}\right) + e^{2k_1a}\left(-2-\frac{7i}{2}\right)\right]}$$

g.) $\psi_3(x) = Fe^{3ik_1x}$

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Probability Current: $J(x,t) = -\frac{i\hbar}{2m}\left[\Psi^*\frac{\partial\Psi}{\partial x} - \Psi\frac{\partial\Psi^*}{\partial x}\right]$

Time component vanishes since only space derivative required for probability density.

$$\psi_3^* = F^*e^{-3ik_1x}$$

$$\frac{d\psi_3}{dx} = 3ik_1Fe^{3ik_1x} \qquad \frac{d\psi_3^*}{dx} = -3ik_1F^*e^{-3ik_1x}$$

$$J = -\frac{i\hbar}{2m}\left[F^*e^{-3ik_1x}\left(3ik_1Fe^{3ik_1x}\right) - Fe^{3ik_1x}\left(-3ik_1F^*e^{-3ik_1x}\right)\right]$$

$$= -\frac{i\hbar}{2m}\left[F^*F(3ik_1) - F^*F(-3ik_1)\right]$$

$$= -i^2\frac{3\hbar k_1\cancel{2}F^*F}{\cancel{2}m}$$

$$\boxed{J = \frac{3\hbar k_1}{m}F^*F}$$

$J \neq \emptyset$ means there is probability flowing from region 1 into region 3.

⑤ a.) The quantum numbers are:

principal quantum number $n = 1, 2, 3 \ldots$

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Orbital quantum number $l = 0, 1, 2, \ldots (n-1)$

magnetic quantum number $m = -l, (-l+1), \ldots 0, \ldots (l-1), l.$

b.) The principal quantum number n determines all others. It gives us the total energy of the Hydrogen atom.

The orbital quantum number l gives the total angular momentum of a given state.

3

The magnetic quantum number m gives the z-component of the angular momentum.

c.) $l = 2, \quad m = -1$

$$Y_{2-1}(\theta, \phi) = B_{2-1} \sin\theta \cos\theta \, e^{-i\phi}$$

$$\int_0^{2\pi} \int_0^{\pi} Y_{2-1}^* Y_{2-1} \sin\theta \, d\theta \, d\phi = 1 \qquad \text{Born's rule.}$$

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$$2\pi \int_0^{\pi} B_{2-1}^2 \sin^3\theta \cos^2\theta \, d\theta = 1$$

$$2\pi B_{2-1}^2 \left[\frac{4}{15} \right] = 1$$

$$B_{2-1}^2 = \frac{15}{8\pi} \quad \text{so} \quad \boxed{B_{2-1} = \sqrt{\frac{15}{8\pi}}}$$

d.) $n = 3$, $\ell = 2$ $\quad R_{32}(r) = A_{32}\left(\frac{r}{a_0}\right)^2 e^{-r/3a_0}$

3

$$P(r) = r^2 R_{n\ell}^* R_{n\ell}$$

$$= r^2 R_{32}^* R_{32} = r^2 A_{32}^2 \left(\frac{r}{a_0}\right)^4 e^{-2r/3a_0} = A_{32}^2 \frac{r^6}{a_0^4} e^{-2r/3a_0}$$

$$\frac{dP}{dr} = \frac{A_{32}^2}{a_0^4}\left(r^6 \left(-\frac{2}{3a_0}\right) e^{-2r/3a_0} + 6r^5 e^{-2r/3a_0} \right) = \emptyset.$$

$$-\frac{2r^6}{3a_0} + 6r^5 = \emptyset$$

$$-2r^6 + 18a_0 r^5 = \emptyset$$

$$-2r + 18a_0 = \emptyset$$

$$\boxed{r = 9a_0}$$

e.)

$$\int_0^\infty P(r)\,dr = 1$$

4

$$\int_0^\infty \frac{A_{32}^2}{a_0^4} r^6 e^{-2r/3a_0}\,dr = \frac{A_{32}^2}{a_0^4}\left[\frac{6!}{\left(\frac{2}{3a_0}\right)^7}\right] = 1$$

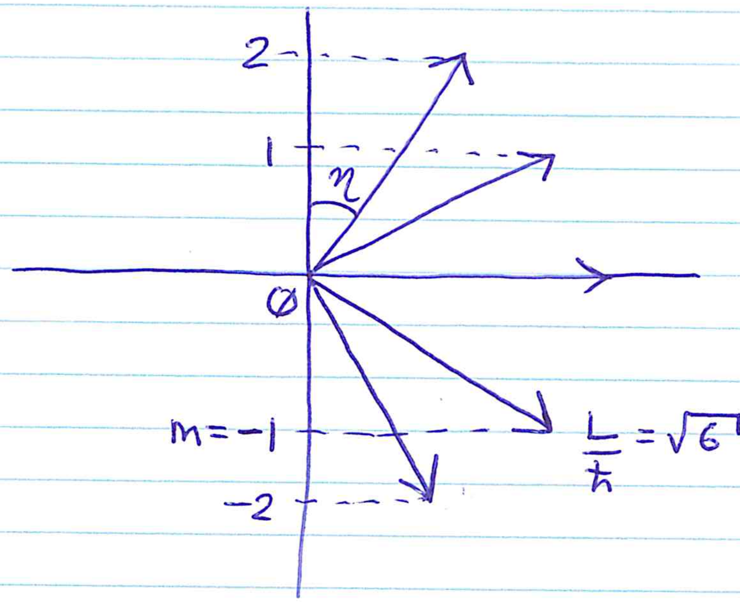
$$A_{32}^2 = \frac{a_0^4\, 2^7}{6!\,(3^7 a_0^7)} = \frac{2^7}{6!\,3^7}\frac{1}{a_0^3} \qquad A_{32} = \sqrt{\frac{2^7}{6!\,3^7}}\,\frac{1}{a_0^{3/2}}$$

$$\frac{2^7}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 3^7} = \frac{2^7}{2\cdot 5\cdot 4\cdot 2\cdot 3^9} = \frac{2^3}{3^9\cdot 5} \quad \text{so} \quad A_{32} = \frac{2\sqrt{2}}{3^3\sqrt{3^3 5}}\,\frac{1}{a_0^{3/2}}$$

$$\boxed{A_{32} = \frac{2\sqrt{2}}{27\sqrt{5}}\,\frac{1}{(3a_0)^{3/2}}}$$

f.) $n=3, \ \ell=2$: $m = -2, -1, 0, 1, 2$

$$\frac{L}{\hbar} = \sqrt{2\cdot 3} = \sqrt{6}$$



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g.) $\cos\eta = \dfrac{m}{\sqrt{\ell(\ell+1)}} = \dfrac{\ell}{\sqrt{\ell(\ell+1)}}$ minimum angle

$$\cos\eta = \frac{2}{\sqrt{6}} = \frac{2}{\sqrt{2}\sqrt{3}}$$

$$\cos\eta = \sqrt{\frac{2}{3}}$$

1

$$\boxed{\eta = \cos^{-1}\left(\sqrt{\frac{2}{3}}\right)}$$