PHYS 2380 - Assignment 6

Final Exam April 27, 2017
7:00 pm, E2-130 EITC

April 25, 2017

1.) The wavefunction for the first excited state of the simple harmonic oscillator is

$$\psi_1(x) = \left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4} x e^{-m\omega x^2/2\hbar} \tag{1}$$

a.) Using the wavefunction ψ_1 above, find the expectation value of the squared position, $< x^2 >$.

b.) Find the expectation value of the squared momentum $< p^2 >$.

c.) Assuming $< x >= 0$ and $< p >= 0$, show that this state obeys the uncertainty principle.

2.) A particle of mass m travels in a circular orbit around a massive attractive centre with a potential that is given by $V(r) = Cr^2$.

a.) Using the equations for circular motion, show that the total energy of such a particle must be given by $2Cr^2$.

b.) Use the Bohr postulate for quantization, $L = pr = mvr = n\hbar$, in combination with the answer to part a to arrive at an expression for the energy associated with the allowed orbits in terms of the mass of the particle, n, and other constants.

3.) A particle in a box with infinitely high walls has a simple wave function

$$\psi(x) = A\sin\left(\frac{\pi x}{a}\right). \tag{2}$$

a.) Find the normalization constant.

b.) Find the probability to find the particle between the left hand wall at $x = 0$ and $a/4$ (ie, what is the probability of finding the particle in the first quarter of the length of the box?)

4.) For the potential function shown in figure 1:

a.) Write down the wave function for a particle with $E > V_0$ in region 1 ($x < -a$), and region 2 ($-a < x < 0$). Provide expressions for the wave number k in each region. Identify the incident and reflected components in these wave functions.

b.) Apply the boundary conditions and determine the amplitudes for all the component waves in terms of one of them.

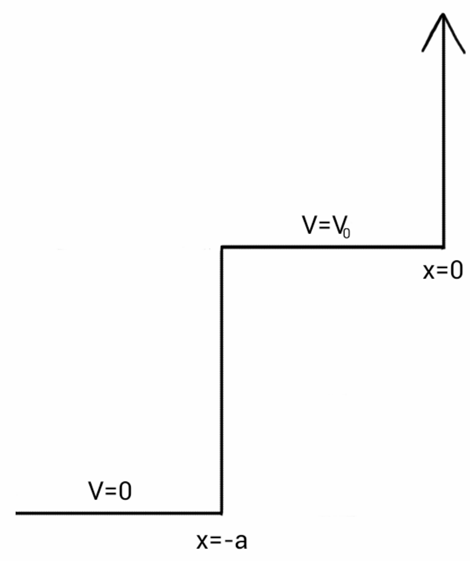

Figure 1: Figure for use with problem 4

c.) From the coefficients determined in part (b), show that the reflection coefficient in region 1 ($x < -a$) is 1.

5.) An electron is confined in an infinite square well potential of width a. You may find it useful to consider such a potential as extending over the region $0 \leq x \leq a$.

a.) Solve the Schrodinger equation for the wave functions of such a particle and derive an expression for the allowed values of its energy.

b.) Calculate the average value for x (where x is the coordinate of the trapped particle).

c.) Calculate the average value for x^2 (where x is the coordinate of the trapped particle).

d.) From these calculate the uncertainty in the position of the particle Δx.

6.) The normalized wave functions for the $n = 2$ state of the hydrogen atom are

$$\psi_{nlm} = R_{nl}(r)\Theta_{lm}(\theta)\Phi_m(\phi) \tag{3}$$

where:

$$R_{20} = \frac{1}{\sqrt{2a_0^3}}\left(1 - \frac{r}{2a_0}\right)e^{-r/2a_0} \tag{4}$$

$$R_{21} = \frac{1}{2\sqrt{6a_0^3}}\left(\frac{r}{a_0}\right)e^{-r/2a_0} \tag{5}$$

and the angular dependence is given by:

$$Y_{11} = \Theta_{11}\Phi_1 = \sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi} \tag{6}$$

$$Y_{10} = \Theta_{10}\Phi_0 = \sqrt{\frac{3}{4\pi}}\cos\theta \tag{7}$$

$$Y_{1-1} = \Theta_{1-1}\Phi_{-1} = \sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi} \tag{8}$$

where a_0 is the Bohr radius.

a.) Calculate the expectation value for r for R_{20} and compare it with the radius predicted by Bohr's theory for this energy level.

b.) Is R_{21} normalized by itself?

c.) Is $Y_{11} = \Theta_{11}\Phi_1$ normalized by itself?

7.) A particle with energy E is incident from the left on the potential function $V(x)$, shown in figure 2, where $V = 0$ for $x \leq 0$ and $V = -V_0 = -3E$ for $x > 0$.

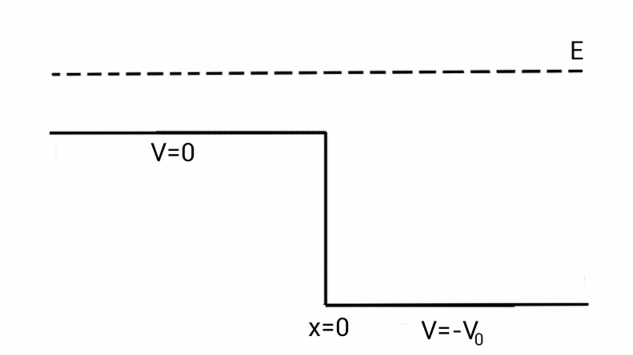

Figure 2: Figure for use with problem 7

a.) Write down the most general "space" wave function in both regions. Should any of the components of these wave functions have zero amplitude? Explain.

b.) Provide expressions for the wave number k in each region. Express one of them in terms of the other.

c.) Apply boundary conditions and determine the amplitudes for all component waves in terms of one of them.

d.) From the coefficients determined in the previous parts, derive an expression for the probability density as a function of x.

e.) Sketch the probability density and locate the maxima, minima and constant levels if any. Assign relative values to the maxima, minima and constant levels.

Using the coefficients for the incident, reflected and transmitted waves from part (d), directly calculate:

f.) The transmission coefficient, T.

g.) The reflection coefficinet, R.

h.) From your results, demonstrate that $T + R = 1$.

8.) The de Broglie wavelength of a particle is defined to be $\lambda = h/p$.

a.) Show that for all energies, large and small, that

$$\lambda = \frac{hc}{E_k\left(1 + 2mc^2/E_k\right)^{\frac{1}{2}}} \tag{9}$$

b.) Show that in the extreme relativistic limit that this expression approaches that for a photon of similar energy.

c.) Show that in the classical limit this expression reduces to $\lambda = h/(mv)$ as expected.

9.) The energy density inside a black body as a function of wavelength λ and temperature T is given by

$$u(\nu) = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/k_bT} - 1} \tag{10}$$

where h is Planck's constant, k_b is the Boltzmann constant and c is the speed of light.

a.) Transform the expression for $u(\nu)$ into an expression for the energy density as a function of wavelength $u(\lambda)$. Recall that the definition of energy density means the power density (W/m^2) contained in a range of frequencies $u(\nu)d\nu$. You results must be consistent with this.

b.) The spectral radiancy of a black body is given by $R(\nu) = \frac{c}{4}u(\nu)$. Integrate the spectral radiancy to arrive at Stefan's Law, which states that the total radiancy of a black body is given by $R_T = \sigma T^4\ W/m^3$. Provide an expression for σ in terms of the constants h, c and k_b.

Formulae:

$$\int_{-\infty}^{\infty} u^2 e^{-u^2} du = \frac{\sqrt{\pi}}{2} \tag{11}$$

$$\int_{-\infty}^{\infty} u^4 e^{-u^2} du = \frac{3\sqrt{\pi}}{4} \tag{12}$$

$$\int_{-\infty}^{\infty} u^4 e^{-u^2} du = \frac{3\sqrt{\pi}}{4} \tag{13}$$

$$\int_0^{n\pi} u \sin^2 u du = \frac{1}{4}\left(u^2 - u\sin 2u - \frac{\cos 2u}{2}\right)\Bigg|_0^{n\pi} \tag{14}$$

$$\int_0^{n\pi} u^2 \sin^2 u du = \left(\frac{u^3}{6} - \frac{1}{4}\left[u^2 - \frac{1}{2}\right]\sin 2u - \frac{u\cos 2u}{4}\right)\Bigg|_0^{n\pi} \tag{15}$$

$$\sin^2 \alpha = \frac{1 - \cos 2\alpha}{2} \tag{16}$$

$$\int_0^\infty x^q e^{-\alpha x} dx = \frac{q!}{\alpha^{q+1}} \tag{17}$$

Relativistic total energy

$$E^2 = p^2 c^2 + m^2 c^4 \tag{18}$$

Relativistic kinetic energy

$$E_k = E - mc^2 \tag{19}$$

$$\int_0^\infty \frac{z^3}{e^z - 1} dz = \frac{\pi^4}{15} \tag{20}$$