PHYS 2380 Quantum Physics 1

Term test March 13 2017, 19:00-21:00, 204 Armes

1.) Planck's law states that the energy density inside a cavity at temperature T is given by

$$u(\lambda) = \left(\frac{8\pi hc}{\lambda^5}\right)\left(\frac{1}{e^{hc/\lambda k_b T} - 1}\right). \tag{1}$$

Show that this leads to Wien's displacement law, which states that the wavelength λ_m where $u(\lambda_m)$ is a maximum and the temperature T are related by

$$\lambda_m T = \frac{hc}{5k_b}. \tag{2}$$

Hint: Assume that at the maximum (i.e., where $\lambda = \lambda_m$), the following relationship is true:

$$\frac{e^{hc/\lambda_m k_b T}}{e^{hc/\lambda_m k_b T} - 1} \approx 1 \tag{3}$$

2.) Suppose the wavefunction of a double slit apparatus is expressed as the sum of two components that correspond to either individual slit. Call the wavefunction of an individual slit Ψ_j and let it consist of a phase component, ϕ_j, and a real amplitude ψ_j, such that

$$\Psi_j = \psi_j \exp(i\phi_j) \tag{4}$$

where both the Ψ_j are solutions of Schrödinger's equation.

a.) Write down the total wavefunction Ψ that describes the wavefunction of both slits ($j = 1, 2$). Why are you able to write the total wavefunction in this way? Justify your answer.

b.) Show that the probability density of the total wavefunction contains an interference term that depends on a simple trig function.

3.) The Copenhagen interpretation of quantum mechanics can be roughly summarized in seven main points. Clearly list them and give quantitative details where possible.

4.) Consider the first two excited states ($n = 2, 3$) of a particle in a box with infinitely high walls. In terms of the ground state energy

$$E_1 = \frac{\hbar^2\pi^2}{2ma^2} \tag{5}$$

the full wavefunctions are

$$\psi_2(x,t) = A\sin\left(\frac{2\pi x}{a}\right)e^{-4iE_1 t/\hbar} \tag{6}$$

$$\psi_3(x,t) = B\cos\left(\frac{3\pi x}{a}\right)e^{-9iE_1 t/\hbar}. \tag{7}$$

a.) Are the wave functions $\psi_2(x,t)$ and $\psi_3(x,t)$ eigenfunctions of the energy operator? Explain your reasoning.

b.) Consider the combination

$$\psi(x) = \frac{1}{\sqrt{a}} \left[\sin\left(\frac{2\pi x}{a}\right) e^{-4iE_1 t/\hbar} + \cos\left(\frac{3\pi x}{a}\right) e^{-9iE_1 t/\hbar} \right] . \tag{8}$$

Is the combined wavefunction $\psi(x)$ an eigenfunction of the energy operator? Explain your reasoning.

c.) For the wavefunction in part b, show that the probability density $P(x)$ is time dependent and real.

d.) Again using the wavefunction in part b, show that

$$< p >= m \frac{\mathrm{d}}{\mathrm{dt}} < x > . \tag{9}$$

5.) Suppose that in some region of space, a particle propagates with energy E greater than the constant potential V_0. Solve the Schrödinger equation to find the wavefunction that describes the particle.

b.) If a particle is in a classically forbidden region, it has energy E less than the constant potential V_0. Solve the Schrödinger equation in this case and find the particle wavefunction.

Physical Constant	Standard Value	Alternative Units
Speed of light	$c = 2.998 \times 10^8$ m/s	
Electron charge	$e = 1.602 \times 10^{-19}$ C	
Boltzmann constant	$k_b = 1.381 \times 10^{-23}$ J/K	8.617×10^{-5} eV/K
Planck's constant	$h = 6.626 \times 10^{-34}$ J.s	4.136×10^{-15} eV.s
	$\hbar = 1.055 \times 10^{-34}$ J.s	6.582×10^{-16} eV.s
Avogadro's number	$N_A = 6.022 \times 10^{23}$ mol	
Stefan-Boltzmann constant	$\sigma = 5.67 \times 10^{-8}$ W /m^2.K^4	
Electron mass	$m_e = 9.11 \times 10^{-31}$ kg	0.511 MeV/c^2
Proton mass	$m_p = 1.673 \times 10^{-27}$ kg	938.3 MeV/c^2
Neutron mass	$m_n = 1.675 \times 10^{-27}$ kg	939.6 MeV/c^2
Mass of the Sun		
Bohr radius	$a_0 = 4\pi\epsilon_0\hbar^2/m_e e^2 = 0.0529$ nm	
Hydrogen ionization energy	13.6 eV	
Coulomb constant	$\frac{1}{4\pi\epsilon_0} = 8.988 \times 10^9$ N.m^2/C^2	

Equations:

Photons:

$$\lambda\nu = c \tag{10}$$

$$E = h\nu = cp_{photon} \tag{11}$$

$$p_{photon} = \frac{h}{\lambda} \tag{12}$$

Blackbody radiation

$$I = \sigma T^4 \tag{13}$$

$$\lambda_m T = 2.898m\dot{K} \tag{14}$$

$$u(\lambda) = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/\lambda k_b T} - 1} \tag{15}$$

$$R(\lambda) = \frac{c}{4}u(\lambda) \tag{16}$$

$$dI = R(\lambda)d\lambda \tag{17}$$

Wave-particle relations:

De Broglie wavelength

$$\lambda = \frac{h}{p} \tag{18}$$

Heisenberg Uncertainty Principle

$$\Delta p_x \Delta x \geq \frac{\hbar}{2} \tag{19}$$

$$\Delta E \Delta t \geq \frac{\hbar}{2} \tag{20}$$

Wave packets:

Momentum:

$$p = \frac{h}{\lambda} = \hbar k \tag{21}$$

Kinetic energy:

$$\frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m} = \hbar\omega \tag{22}$$

Group velocity:

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{dk}} \tag{23}$$

Phase velocity:

$$v_p = \frac{\omega}{k} \tag{24}$$

Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x,t)\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t} \tag{25}$$

Time independent Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi(x)}{\mathrm{dx}^2} + V(x)\psi(x) = E\psi(x) \tag{26}$$

Bohr model energy levels:

$$E_n = -\frac{Z_{eff}^2 m e^4}{32\pi^2\epsilon_0^2\hbar^2}\frac{1}{n^2} = -\frac{13.6 Z_{eff}^2}{n^2} eV \tag{27}$$

Trig identities:

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \tag{28}$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\cos\beta \tag{29}$$

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha \tag{30}$$

$$e^{i\alpha} = \cos\alpha + i\sin\alpha \tag{31}$$

$$\cos\alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2} \tag{32}$$

$$\sin\alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i} \tag{33}$$

$$\cosh\alpha = \frac{e^{\alpha} + e^{-\alpha}}{2} \tag{34}$$

$$\sinh\alpha = \frac{e^{\alpha} - e^{-\alpha}}{2} \tag{35}$$

Useful integrals:

$$\int_{-a/2}^{a/2} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi x}{a}\right) dx = \frac{4a}{3\pi} \tag{36}$$

$$\int_{-a/2}^{a/2} \cos\left(\frac{\pi x}{a}\right) \cos\left(\frac{2\pi x}{a}\right) dx = \frac{2a}{3\pi} \tag{37}$$

$$\int_{-a/2}^{a/2} \sin\left(\frac{2\pi x}{a}\right) \sin\left(\frac{3\pi x}{a}\right) dx = \frac{4a}{5\pi} \tag{38}$$

$$\int_{-a/2}^{a/2} \cos\left(\frac{2\pi x}{a}\right) \cos\left(\frac{3\pi x}{a}\right) dx = \frac{6a}{5\pi} \tag{39}$$

$$\int_{-a/2}^{a/2} x \sin\left(\frac{\pi x}{a}\right) \cos\left(\frac{2\pi x}{a}\right) dx = -\frac{10a^2}{9\pi^2} \tag{40}$$

$$\int_{-a/2}^{a/2} x \sin\left(\frac{2\pi x}{a}\right) \cos\left(\frac{3\pi x}{a}\right) dx = -\frac{24a^2}{25\pi^2} \tag{41}$$

The following relations hold for any β, γ:

$$\int_{-a/2}^{a/2} \sin(\beta x) \cos(\gamma x)\, dx = 0 \tag{42}$$

$$\int_{-a/2}^{a/2} x \cos(\beta x) \cos(\gamma x)\, dx = 0 \tag{43}$$

$$\int_{-a/2}^{a/2} x \sin(\beta x) \sin(\gamma x)\, dx = 0 \tag{44}$$