UNIVERSITY OF MANITOBA

April 27^{th}, 2017
COURSE NO.: PHYS 2380
TIME: 3 hours

Final Exam
EXAM: Quantum Physics 1
NO. OF PAGES: 7

Examiner: A. Rogers

Instructions:

- Answer all questions.
- Start each question on a new page.
- Show all of your steps in arriving at the solution. Do not just state the answer. State your arguments clearly. Indicate any assumptions that you make.
- Include a sentence or two at the end of each problem summarizing your solution.

1.) Semi-classical dynamics (10 marks)

Suppose that a particle of mass m travels in a circular orbit around a massive center with a potential given by $V(r) = Cr^2$ where C is a constant.

a.) Using the equations of circular motion, show that the total energy of such a particle must be given by $2Cr^2$.

b.) Use the Bohr postulate for quantization $L = pr = mvr = n\hbar$ in combination with the answer to part a to arrive at an expression for the energy associated with the allowed orbits in terms of the mass of the particle, n, and the other constants.

2.) A mystery potential (12 marks)

The wavefunction that corresponds to some unknown potential $V(x)$ is given by

$$\psi(x) = Ae^{-x^2/2a^2} \tag{1}$$

which is a solution of the Schrodinger equation with energy $E = \hbar^2/2ma^2$ (A and a are constants).

a.) What is the potential that the particle experiences?

b.) Does the result in part a correspond to any classical potential? If so, specify any constants the potential depends on.

3.) Properties of the angular momentum operators (14 marks)

In this problem we will use the separable hydrogen wavefunction

$$\psi_{nlm}(r, \theta, \phi) = R_{nl}(r)\Theta_{lm}(\theta)\Phi_m(\phi). \tag{2}$$

a.) The azimuthal part of the separable hydrogen wavefunction satisfies the differential equation

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -m^2. \tag{3}$$

Solve this equation for $\Phi(\phi)$. With no barriers present, you may discard any reflected component. Show that your solution is an eigenfunction of the z-component of the angular momentum operator $L_{z,op}$ and find the corresponding eigenvalue.

b.) The polar part of the hydrogen wavefunction obeys the differential equation

$$\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta_{lm}}{d\theta}\right) - m^2\Theta_{lm} + l(l+1)\sin^2\theta\Theta_{lm} = 0. \tag{4}$$

Let the squared angular momentum operator L^2_{op} operate on the separable wavefunction. Use the DE above, and the solution from part a to show that the separable hydrogen wavefunction ψ_{nlm} is an eigenfunction of L^2_{op}, and find the corresponding eigenvalue. *Hint: You do not need to explicitly solve the DE above for Θ_{lm} to complete this problem.*

4.) The barrier potential (20 marks)

A particle is incident from the left on a potential barrier shown in figure 1.

a.) Write down the most general forms for the "space-part" of the wave function for a particle with energy $E = 9V_0$ (where V_0 is a constant) in region 1 ($x < 0$), region 2 ($0 \leq x \leq a$) and region 3 ($x > a$). Note that you do **not** have to derive these wavefunctions! Just write them down.

b.) Provide expressions for the wave number k in each region (you may find it helpful to express the wave numbers as multiples of the wave number in region 1).

c.) Identify the forward and reflected components of the propagating wave functions. Are any of these terms obviously zero?

d.) What are the corresponding "time-parts" of the wave function in each region?

e.) Apply the boundary conditions required, at $x = 0$ and $x = a$ to arrive at 4 equations that relate the amplitudes of the component waves.

f.) Solve the equations from part e to express the amplitude of the component waves in terms of the amplitude of the wave function in region 3 ($x > a$).

g.) Use the wave function in region 3 to calculate the probability current entering this region. Does any probability flow through the barrier?

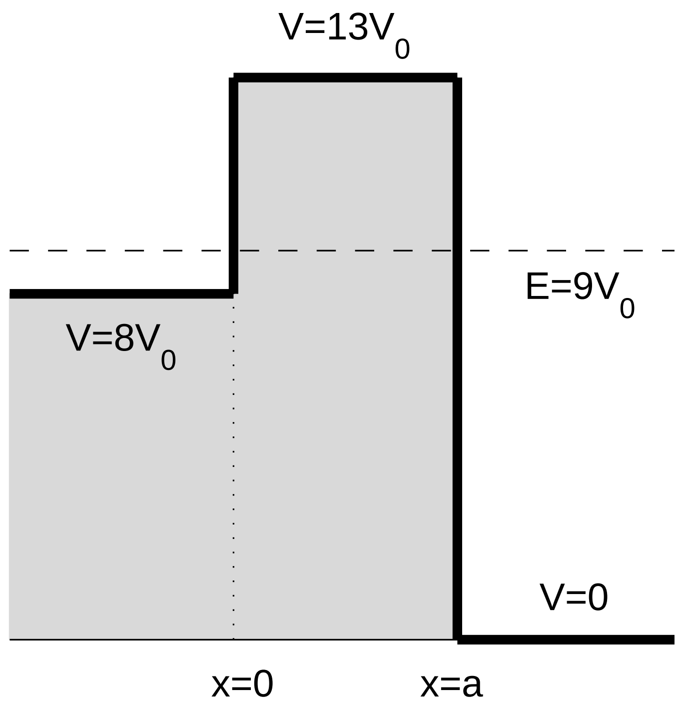

Figure 1: The potential barrier used in problem 4.

5.) The hydrogen atom (20 marks)

a.) What are the quantum numbers for the stationary states of the hydrogen atom? What values can these quantum numbers assume?

b.) What physical properties are associated with each quantum number?

c.) Normalize the spherical harmonic state with $l = 2$, $m = -1$ (given on the formula sheet).

d.) Using the $n = 3$, $l = 2$ radial wavefunction from the formula sheet, write the radial probability density and find the position where the probability density is a maximum.

e.) Using the probability density from part d, find the normalization constant of the $n = 3$, $l = 2$ radial wavefunction.

f.) For the $n = 3$, $l = 2$ state, draw a figure showing all possible orientations of the angular momentum vector L, and indicate the configuration of the $m = -1$ state.

g.) What is the smallest angle that L can make with respect to the z-axis? Indicate this on the figure and calculate this angle.

THE END!

Physical Constant	Standard Value	Alternative Units
Speed of light	$c = 2.998 \times 10^{8}$ m/s	
Electron charge	$e = 1.602 \times 10^{-19}$ C	
Boltzmann constant	$k_b = 1.381 \times 10^{-23}$ J/K	8.617×10^{-5} eV/K
Planck's constant	$h = 6.626 \times 10^{-34}$ J.s	4.136×10^{-15} eV.s
	$\hbar = 1.055 \times 10^{-34}$ J.s	6.582×10^{-16} eV.s
Avogadro's number	$N_A = 6.022 \times 10^{23}$ mol	
Stefan-Boltzmann constant	$\sigma = 5.67 \times 10^{-8}$ W /m^2.K^4	
Electron mass	$m_e = 9.11 \times 10^{-31}$ kg	0.511 MeV/c^2
Proton mass	$m_p = 1.673 \times 10^{-27}$ kg	938.3 MeV/c^2
Neutron mass	$m_n = 1.675 \times 10^{-27}$ kg	939.6 MeV/c^2
Mass of the Sun		
Bohr radius	$a_0 = 4\pi\epsilon_0\hbar^2/m_e e^2 = 0.0529$ nm	
Hydrogen ionization energy	13.6 eV	
Coulomb constant	$\frac{1}{4\pi\epsilon_0} = 8.988 \times 10^{9}$ N.m^2/C^2	

The wave functions of hydrogen-like atoms are

$$\psi_{nlm}(r,\theta,\phi) = R_{nl}(r)Y_{lm}(\theta,\phi) \tag{5}$$

with spherical harmonics

$$Y_{lm}(\theta,\phi) = \Theta_{lm}(\theta)\Phi_m(\phi). \tag{6}$$

Radial Wave Functions with arbitrary constants A_{nl}:

$$R_{10}(r) = A_{10}e^{-r/a_0} \tag{7}$$

$$R_{20}(r) = A_{20}\left(1 - \frac{r}{2a_0}\right)e^{-r/2a_0} \tag{8}$$

$$R_{21}(r) = A_{21}\left(\frac{r}{a_0}\right)e^{-r/2a_0} \tag{9}$$

$$R_{30}(r) = A_{30}\left(1 - 2\frac{r}{3a_0} - 2\frac{r^2}{27a_0^2}\right)e^{-r/3a_0} \tag{10}$$

$$R_{31}(r) = A_{31}\left(1 - \frac{r}{6a_0}\right)e^{-r/3a_0} \tag{11}$$

$$R_{32}(r) = A_{32}\left(\frac{r}{a_0}\right)^2 e^{-r/3a_0} \tag{12}$$

For normalized spherical harmonics, the radial probability density is:

$$P(r) = r^2 R_{nl}^* R_{nl} \tag{13}$$

Spherical Harmonics with arbitrary constants B_{lm}:

$$Y_{10} = B_{10}\cos\theta \tag{14}$$

$$Y_{11} = B_{11}\sin\theta e^{i\phi} \tag{15}$$

$$Y_{1-1} = B_{1-1}\sin\theta e^{-i\phi} \tag{16}$$

$$Y_{20} = B_{20}\left(3\cos^2\theta - 1\right) \tag{17}$$

$$Y_{21} = B_{21}\sin\theta\cos\theta e^{i\phi} \tag{18}$$

$$Y_{2-1} = B_{2-1}\sin\theta\cos\theta e^{-i\phi} \tag{19}$$

$$Y_{22} = B_{22}\sin^2\theta e^{2i\phi} \tag{20}$$

$$Y_{2-2} = B_{2-2}\sin^2\theta e^{-2i\phi} \tag{21}$$

Equations:

Photons:

$$\lambda\nu = c \tag{22}$$

$$E = h\nu = cp_{photon} \tag{23}$$

$$p_{photon} = \frac{h}{\lambda} \tag{24}$$

Blackbody radiation

$$I = \sigma T^4 \tag{25}$$

$$\lambda_m T = 2.898m\dot{K} \tag{26}$$

$$u(\lambda) = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/\lambda k_b T} - 1} \tag{27}$$

$$R(\lambda) = \frac{c}{4}u(\lambda) \tag{28}$$

$$dI = R(\lambda)d\lambda \tag{29}$$

Mechanics:
Force - Potential relationship:

$$F = -\frac{dV(x)}{dx} \tag{30}$$

Centripetal force:

$$F = \frac{mv^2}{r} \tag{31}$$

Classical angular momentum:

$$L = r \times p = rp\sin\theta = mvr\sin\theta \tag{32}$$

where θ is the angle between the position and momentum vectors.
Relativistic total energy

$$E^2 = p^2c^2 + m^2c^4 \tag{33}$$

Relativistic kinetic energy

$$E_k = E - mc^2 \tag{34}$$

Morse potential:

$$V(x) = C\left(1 - e^{a(x-x_0)}\right)^2 \tag{35}$$

Simple harmonic oscillator potential:

$$V(x) = \frac{1}{2}Cx^2 = 2\pi^2 m\nu^2 x^2 \tag{36}$$

with frequency of oscillation

$$\nu = \frac{1}{2\pi}\sqrt{\frac{C}{m}} \tag{37}$$

Coulomb potential:

$$V(r) = \frac{-Ze^2}{4\pi\epsilon_0}\frac{1}{r} \tag{38}$$

Yukawa potential:

$$V(x) = -C\frac{e^{-ax}}{x} \tag{39}$$

Wave-particle relations:

De Broglie wavelength

$$\lambda = \frac{h}{p} \tag{40}$$

Heisenberg Uncertainty Principle

$$\Delta p_x \Delta x \geq \frac{\hbar}{2} \tag{41}$$

$$\Delta E \Delta t \geq \frac{\hbar}{2} \tag{42}$$

Wave packets:

Momentum:

$$p = \frac{h}{\lambda} = \hbar k \tag{43}$$

Kinetic energy:

$$\frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m} = \hbar\omega \tag{44}$$

Group velocity:

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{dk}} \tag{45}$$

Phase velocity:

$$v_p = \frac{\omega}{k} \tag{46}$$

Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x,t)\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t} \tag{47}$$

Time independent Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi(x)}{\mathrm{dx}^2} + V(x)\psi(x) = E\psi(x) \tag{48}$$

Bohr model energy levels:

$$E_n = -\frac{\mu Z^2 e^4}{2(4\pi\epsilon_0)^2\hbar^2}\frac{1}{n^2} = -\frac{13.6Z^2}{n^2}eV \tag{49}$$

Probability current:

$$J = -\frac{i\hbar}{2m}\left[\Psi^*\frac{\partial\Psi}{\partial x} - \Psi\frac{\partial\Psi^*}{\partial x}\right] \tag{50}$$

Angular momentum operators:

$$L^2_{op} = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \tag{51}$$

$$L_{z,op} = -i\hbar\frac{\partial}{\partial\phi} \tag{52}$$

Trig identities:

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \tag{53}$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\cos\beta \tag{54}$$

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha \tag{55}$$

$$\sin^2(\alpha) = \frac{1-\cos(2\alpha)}{2} \tag{56}$$

$$\cos^2(\alpha) = \frac{1+\cos(2\alpha)}{2} \tag{57}$$

$$e^{i\alpha} = \cos\alpha + i\sin\alpha \tag{58}$$

$$\cos\alpha = \frac{e^{i\alpha}+e^{-i\alpha}}{2} \tag{59}$$

$$\sin\alpha = \frac{e^{i\alpha}-e^{-i\alpha}}{2i} \tag{60}$$

$$\cosh\alpha = \frac{e^{\alpha}+e^{-\alpha}}{2} \tag{61}$$

$$\sinh\alpha = \frac{e^{\alpha}-e^{-\alpha}}{2} \tag{62}$$

Useful integrals:

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi} \tag{63}$$

$$\int_{-\infty}^{\infty} x^2 e^{-x^2}dx = \frac{\sqrt{\pi}}{2} \tag{64}$$

$$\int_{-\infty}^{\infty} x^4 e^{-x^2}dx = \frac{3\sqrt{\pi}}{2} \tag{65}$$

$$\int_{0}^{\infty} x^q e^{-\alpha x}dx = \frac{q!}{\alpha^{q+1}} \tag{66}$$

$$\int_0^a \sin x\cos x dx = \frac{1}{2}\sin^2 x\Big|_0^a \tag{67}$$

$$\int_0^a \cos^3 x dx = \frac{1}{3}\sin x\left(\cos^2 x + 2\right)\Big|_0^a \tag{68}$$

$$\int_0^a \sin^3 x dx = -\frac{1}{3}\cos x\left(\sin^2 x + 2\right)\Big|_0^a \tag{69}$$

$$\int_0^a \sin^3 x\cos^2 x dx = -\frac{1}{5}\cos^3 x\left(\sin^2 x + \frac{2}{3}\right)\Big|_0^a \tag{70}$$

$$\int_0^a \sin^n x\cos x dx = \frac{\sin^{n+1} x}{n+1}\Big|_0^a \tag{71}$$

with $n > 0$.

$$\int_0^a \cos^n x\sin x dx = -\frac{\cos^{n+1} x}{n+1}\Big|_0^a \tag{72}$$

with $n > 0$.